§ 3. Algorithm of Resolution

Now we are ready to prove

$$\text{Ord II in dim} \le n-1 \xRightarrow{T_1} \text{Ord I in dim } n \xRightarrow{T_2} \text{Ord II in dim } n.$$

T_1: We start from (X, I, E) $E = (E^1, \cdots, E^s)$ (div smooth) $\text{maxord } I \le m$

If $\text{maxord} < m$, the process is trivial. We assume $\text{maxord } I = \boxed{m}$

Step 1: Construct $\Pi_1 : (X_{r_1}, I_{r_1}, E_{r_1}) \to \cdots \to (X, I, E)$ s.t.

$\text{Supp}\, \Pi_{1*}^{-1} E \cap \text{Supp}\,(I_{r_1}, m) = \phi$.

Consider the equivalent ideal $\boxed{C(I)}$ now

1.1. Let Z_0 be the union of all irr comp of E' contained in $\text{Supp}(I, m)$.

Blow up Z_0, we have (for irr comp E^{1k} in E' for example) $\Pi_0 : X_1 \to X$

$$\text{maxord}_{E^{1k}} I = m \Rightarrow \text{max}_{\Pi_{0*}^{-1} E^{1k}} \Pi_0^{*-1} I \le m - m = 0.$$

$\Rightarrow \Pi_{0*}^{-1} E^{1k}$ and $\text{supp}(I_{r_1}, m)$ are disjoint.

1.2. Now, set $S = E^1$, $E_S = (E - E^1)|_S = (\emptyset, E^2|_S, \cdots, E^s|_S)$

consider $(S, I|_S, m, E|_S)$ apply Ord II in dim $\le n-1$, we get

$BO_1 : (S_{E_1}, I_{E_1}|_{S_{E_1}}, m, E_{S_{E_1}})$ lift $(X_{r_{E_1}}, I_{r_{E_1}}, m, E_S) \to \cdots \to (\cdots)$

s.t. $\text{supp}(I_{r_{E_1}}, m) \cap \underset{\underset{\Pi_{E_1*}^{-1} E_1}{\shortparallel}}{\Pi_{E_1*}^{-1} S} = \phi$

Inductively, we have $\Pi_1 : (\cdots) \to \cdots (\cdots)$ s.t. $\text{Supp}\, \Pi_{1*}^{-1} E \cap \text{Supp}\,(I_{r_1}, m) = \phi$

Functoriality follows from previous remark.

Step 2: Start from $J = H(C(I_{r_1}))$, $Y = X_{r_1}$, $F = (\Pi_*^{-1} E, E^{s+1}, \cdots, E^{s+r_1})$ (excep.)

(Y, J, m, F).

$\forall y \in \text{Supp}(J, m)$, $\exists U_y$. $h \in T(J)$ Since in step 1, all blow-up is snc,

we can take h s.t. H_h snc with F ($\Pi_*^{-1} E$ away from $\text{supp}(I, m)$). ~~X~~

locally consider $(H_h, J|_{H_h}, m, F|_{H_h})$

Apply Ord II in dim $< n$ and lift it and globalized it to

$\Pi_2 : (X_{r_2}, I_{r_2}, E_{r_2}) \to (X_{r_1}, I_{r_1}, E_{r_1})$

s.t. $\text{Supp}\,(I_{r_2}, m) = \phi$!

(Note, all blow up seq is also for $C(I)$ and I, $TH(C(I)) = C(I)$ ☆ ✓)

Functoriality follows from previous rem.

□.

T2: We start with a marked triple (X, I, m, E) $E=(E^1,\cdots,E^s)$.

Step 0:

We may write $I=N(I)M(I)$, $M(I)=\mathcal{O}_X(-\sum c_i E_i)$ and $\operatorname{Supp} N(I)$ does not contain any of E^i!

Rem: if $E=\phi$, $M(I)=\phi$ and $I=N(I)$.

Step 1: Write $(N(I),1)+(I,m)=(J, s)$ here s a number.

Write $m_J=\text{maxord } J$, Run Ord I to (X, J, E) with m_J.

we get (X_1, J_1, E_1) s.t. $\text{maxord } J_1 = m_{J_1} < m_J$.

(Note, $\operatorname{Supp}(J, m_J)_k \subseteq \operatorname{Supp}(J, s)_k = \operatorname{Supp}(N(I),1)_k \cap \operatorname{Supp}(I, m)_k$).

Inductively we get

(X_r, J_r, E_r) s.t. $\text{maxord } J_r < s$.

this implies $\phi = \operatorname{Supp}(J, s)_r = \operatorname{Supp}(N(I),1)_r \cap \operatorname{Supp}(I m)_r$

$$\operatorname{Supp}(N(I),1)_r = \operatorname{Supp}(N(I)_r).$$

To sum up, we have $(X_r, I_r, m, E_r) \to \cdots \to (X, I, m, E)$

s.t. $N(I)_r \cap \operatorname{Supp}(I, m)_r = \phi$.

Rem: $N(I)_r$ $N(I_r)$ differs by some exceptional comps E^k $(k>s)$

$M(I)_r$ $M(I_r)$ and is contained in $M(I_r)$.

Step 2: $I=M(I)=\mathcal{O}_X(-\sum a_j E^j)$ $E=(E^1, E^2,\cdots, E^s,\cdots)$.

2.1 Sub $\{E^1, E^2,\cdots, E^s,\cdots\}$ has a lexicographic order.

$(x\cdots\cdots)$.

2.2. $\forall x \in X$, set $p(x)=(\{E^{j_1},\cdots,E^{j_k}\})$ the maximal subset (in above order) satisfying

(1) E^{j_i} pass x $\forall$ $1\le i\le k$.

(2) $\sum_k a_{j_i} \ge m$ (3) $a_{j_1}+\cdots+\hat{a}_{j_i}+\cdots+a_{j_k} < m$.

$D_{p(x)} = \bigcap_{i=1} E^{j_i}$, and it is the focus that is a maximal component of $\operatorname{Supp}(I, m)$.

$\nu = (\text{maxord } I$, member of maximal comp of $\operatorname{Supp}(I, m)$ attain maxord$)$

$= (m, n)$

Each time, we blow up $D_{p(x)}$, $D_{p(x)} = \{x_{j_1} = \cdots = x_{j_K} = 0\}$.

$\forall x \in D_{p(x)}$, since $\sum_{i=1}^{K} a_{j_i} - m < a_{j_i}$ $\forall\, 1 \le i \le K$,

$\underline{I_x = \prod_t x_t^{a_t}}$

$\qquad\qquad a_{j_1} + \cdots + \widehat{a_{j_i}} + \cdots + a_{j_K} - m + a_{j_i}$

and $\underline{\pi_*^{-1} I_x = \prod_{t \neq j_i} x_t^{a_t} \cdot x_{j_i}^{\underline{\sum a_{j_i} - m}}}$ $< a_{j_i}$
in cord x_{j_i}

$\Rightarrow \operatorname{ord}_y \pi_*^{-1}(I) < \operatorname{ord}_x I$ $\quad \forall\ \pi(y) = x$.

$\Rightarrow$ ν decrease strictly in the lexi-cographic order $\Rightarrow$ the procedure terminates with $\nu = (<m, *)$. $\Rightarrow$ maxord $I_r < m$ eventually.

Functoriality follows from the sm invariant property of $p(x)$ and ν. $\square$.

OCE: Commute resp to closed embedding or Ord II $E = \phi$, $m = 1$

I_S

$\tau: S \to X$

Now, $\tau_* I_S$ in X we have locally
$\tau_* I_S$ is adding some smooth element $\{u\}$.

$\operatorname{Ord}(\tau_* I_S, 1) = 1$ $\quad T(\tau_* I_S) = \tau_* I_S \Leftarrow$ sm element.

Recall T_1: We restrict to sm hysuf in T to do induction.
so for this case, we just restrict $\tau_* I_S$ to $\{u\}$ and exactly get I, the procedure commute with closed embedding.

$\square$