§1. Marked ideals and ord reductions.

We reduce P II to an inductive order reduction process in this section, and we introduce the "marked ideals" that play an important role in the proces.

Def 1.1 (Order) Let X be a smooth variety, $0 \neq I \subset \mathcal{O}_X$ ideal sheaf, For a point $x \in X$ (not necessarily closed), we define

$$\mathrm{Ord}_x I = \max\{ r : m_x^r \mathcal{O}_{x,X} \supset I \cdot \mathcal{O}_{x,X} \}.$$

Rem: ① Ord_x is constructible, upper-semi-continuous on X.

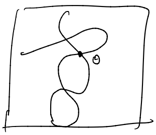
② the maximal order of I along subvar $Z \subset X$ is

$$\text{max-ord}_Z I = \max\{ \mathrm{ord}_z I, z \in Z \}.$$

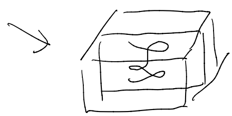
when $Z = X$, we may omit Z.

③ If $V(I)$ is contained in a smooth hyperplane, then $\mathrm{ord}_x I = 1$ $\forall x \in \mathrm{Supp}\, I$. (In this case, $\exists u \in I$, $\mathrm{ord}_x u = 1$ $\forall x \in \mathrm{supp}\, u$).

plane



$\mathrm{ord}_o I_C = 2$ as plane curve



$I_C' = (I_C, u)$ $\quad \mathrm{ord}_x I_C' = 1$.

Def 1.2 (Marked ideals) A marked ideal on a sm var X is a pair (I, m) where $I \subset \mathcal{O}_X$ is an ideal sheaf and m is a natural number. The support of a marked ideal (I, m) is defined as

$$\mathrm{Supp}\,(I, m) = \{ x \in X \mid \mathrm{Ord}_x I \geq m \}.$$

Rem: $\mathrm{Supp}\,(I, 1) = \mathrm{Supp}\,(I)$, $\mathrm{Supp}\,(I, m)$ is closed.

Def 1.3 (Birational transform of Ideals and Marked ideals)

Let $0 \neq I \subset \mathcal{O}_X$ be an ideal sheaf for a smooth variety X, There is a unique largest div $\mathrm{Div}(I)$ s.t. $I \subset \mathcal{O}_X(-\mathrm{Div}(I))$. We may write

$$I = \mathcal{O}_X(-\mathrm{Div}(I)) \cdot I_{\mathrm{codim} \geq 2} \qquad I_{\mathrm{codim} \geq 2} = I \cdot \mathcal{O}_X(\mathrm{Div}(I))$$

Let $f:\tilde{X}\to X$ be a smooth blow-up with center $Z\subset X$ and exceptional div $f^{-1}(Z) := E$. We define the bir transform for I as

$$f_*^{-1}I = \mathcal{O}_{\tilde{X}}(\operatorname{Ord}_{\eta_Z}(I)\cdot E)\cdot f^{-1}I\cdot\mathcal{O}_{\tilde{X}}.$$

We define the bir trans for marked ideal as

$$f_*^{-1}(I,m) = \left(\mathcal{O}_{\tilde{X}}(mE)\cdot f^{-1}I\cdot\mathcal{O}_{\tilde{X}},\, m\right)$$

Remark: ① In application, we require $Z\subseteq \operatorname{Supp}(I,m)$, and the bir trans for marked ideal is well defined in this case.

② The exceptional div here is different from the usual ones. If $f:\tilde{X}\to X$ is a trivial blow-up, then f is Id, $E=Z$. And in this case, $f_*^{-1}I_Z = \mathcal{O}_{\tilde{X}}(E)\cdot\mathcal{O}_{\tilde{X}}(-E) = \mathcal{O}_{\tilde{X}}$

③ f is empty blow-up, then $f_*^{-1}I = I$.

④ $(g^{\in I}, m)$ is called a marked function. And in local coordinate $(x_1,\cdots x_n)$, $Z := (x_1=\cdots=x_r=0)$, blow up Z, the bir transform of (f,m) is

$$f_*^{-1}(g,m) = \left(y_r^{-m}g(y_1 y_r,\cdots,y_{r-1}y_r, y_r, x_{r+1},\cdots,x_{r+n}),m\right)$$ in the local chart resp to x_r ($f_*^{-1}g = y_r^{-\operatorname{ord}_Z g}\, g(\cdots)$)

(I,m) marked ideal,

⑤ Let $Z\subset X$ with $\operatorname{Ord}_{\eta_Z} I = \max\operatorname{ord} I = m$, $\pi: Bl_Z X\to X$ then $\max\operatorname{ord}\pi_*^{-1}I \leq \max\operatorname{ord} I$.

thus $\operatorname{ord}_x f = m$ by semi-cont. $x=f(y)$.

Semi-continuity: $\forall x\in Z$, $\operatorname{ord}_x I = m$. $\forall\ y\in\pi^{-1}(Z)$, $\exists f\in I$ s.t. $\operatorname{ord}_Z f = m$. $\pi_*^{-1}f = y_r^{-m}f(y_1 y_r,\cdots)$

$\Rightarrow \operatorname{ord}_y \pi_*^{-1}f \leq \operatorname{ord}_y f(y_1 y_r\cdots) - m \leq 2m-m = m$. $\Rightarrow \operatorname{ord}_y \pi_*^{-1}I \leq \operatorname{ord}_y \pi_*^{-1}f \leq m \Rightarrow \max\operatorname{ord}\pi_*^{-1}I\leq m$.

⑥ $Z\subsetneq H\subset X$, where H is a hypersurface, $I\subseteq\mathcal{O}_X$ $I|_H\neq 0$. $Z\subseteq\operatorname{Supp}(I,m)$.

$\pi: Bl_Z X\to X$ $\quad \pi|_H\ Bl_Z H\to H$

$(\pi_H)_*^{-1}(I|_H) \supset (\pi_*^{-1}I)|_{Bl_Z H}$ $\quad (\pi_H)_*^{-1}(I|_H, m) = (\pi_*^{-1}(I,m))|_{Bl_Z H}$

(When we do restriction on H, ord may increase, but when we assign an order as in marked ideal, everything is fine)

⑦ $f: Y\to X$ smooth, $f(y)=x$, $I\subseteq\mathcal{O}_X$ then

$$\operatorname{Ord}_y f^{-1}I\cdot\mathcal{O}_Y = \operatorname{Ord}_x I \quad (\text{Check for étale, for } \times\mathbb{A}^n).$$

⑧ The bir trans for ideals and marked ideals are only defined for Seq of sm blow-ups.

Def-Prop 1.4 (Arithmetic Operation on Marked ideals)

Let (I_1, m_1), (I_2, m_2) be two marked ideals on sm var X, we introduce the following

$(I_1, m_1)\cdot(I_2, m_2) = (I_1 I_2, m_1+m_2)$, $\sum_i^n (I_i, m_i) = (\sum_i I_i^{c_i}, \mathrm{lcm}(m_1, \cdots) m_n))$

here $c_i = \mathrm{lcm}(m_1, \cdots, m_n)/m_i$. ($m_i \neq 0$ above). (I_i, m_i) marked ideal on X.

We have the following basic properties.

(1) $\mathrm{Supp}(\sum_i^n (I_i, m_i)) = \bigcap_{i=1}^n \mathrm{Supp}(I_i, m_i)$.

(2) $\mathrm{Supp}(I_1, m_1) \cap \mathrm{Supp}(I_2, m_2) \subseteq \mathrm{Supp}((I_1, m_1)\cdot(I_2, m_2))$

(3) $\mathrm{Supp}(I^c, cm) = \mathrm{Supp}(I, m)$

(4) Let $\pi: Y = Bl_Z \to X$ be a smooth blow-up for $Z \subseteq \mathrm{Supp}(I_1, m_1) \cap \mathrm{Supp}(I_2, m_2)$

we have $\pi_*^{-1}[(I_1, m_1) + (I_2, m_2)] = \pi_*^{-1}(I_1, m_1) + \pi_*^{-1}(I_2, m_2)$

$\pi_*^{-1}[(I_1, m_1)\cdot(I_2, m_2)] = \pi_*^{-1}(I_1, m_1)\cdot\pi_*^{-1}(I_2, m_2)$.

Remark: Here $\sum(I_i, m_i)$ is in fact not even associate operation, it is just a formal notation !

(1) $\forall x \in \bigcap_i \mathrm{Supp}(I_i, m_i)$, $\mathrm{ord}_x I_i^{c_i} \geq \mathrm{lcm}(m_1, \cdots, m_n)$

$\Rightarrow \mathrm{ord}_x I_1^{c_1} + \cdots + I_n^{c_n} \geq \mathrm{lcm} \Rightarrow \bigcap_i \mathrm{Supp}(I_i, m_i) \subseteq \mathrm{Supp}(\sum(I_i, m_i))$

$\forall x \in \mathrm{Supp}(\sum_i (I_i, m_i))$, if $\exists f_i \in I_i$ s.t. $\mathrm{ord}_x f_i < m_i$

$\Rightarrow \mathrm{ord}_x(\sum(I_i, m_i)) < c_i m_i \Rightarrow \Leftarrow$. $\Rightarrow \mathrm{ord}_x I_i \geq m_i$ $\forall i, \forall x$. $\square$.

(2) follows from definition.

(3) by (2) we have $\mathrm{Supp}(I, m) \subseteq \mathrm{Supp}(I^c, cm)$.

$\forall x \in \mathrm{Supp}(I^c, cm)$, assume $\mathrm{ord}_x f < m$ for some $f \in I$

$\Rightarrow \mathrm{ord}_x f^c < cm \Rightarrow \mathrm{ord}_x I^c < cm \Rightarrow \Leftarrow$. $\square$.

(4).

$\pi_*^{-1}((I_1, m_1)(I_2, m_2)) = \pi_*^{-1}(I_1 I_2, m_1+m_2)$

$= \pi^{-1}(I_1 I_2)\cdot O_Y \cdot O_Y((m_1+m_2)E)$

$= \pi^{-1}(I_1)\cdot O_Y \cdot O_Y(m_1 E) \cdot \pi^{-1}(I_2) O_Y \cdot O_Y(m_2 E)$

$= \pi_*^{-1}(I_1, m_1) \cdot \pi_*^{-1}(I_2, m_2)$

$Y = Bl_Z X$.

$\downarrow$

$Z \subset X$

$Z \subseteq \bigcap_i \mathrm{Supp}(I_i, m_i)$

$\pi_*^{-1}((I_1, m_1) + (I_2, m_2)) = \pi_*^{-1}(I_1^{c_1} + I_2^{c_2}, \mathrm{lcm}(m_1, m_2))$

$= \pi^{-1} I_1^{c_1} \cdot O_Y \cdot O_Y(c_1 m_1) + \pi^{-1} I_2^{c_2} \cdot O_Y \cdot O_Y(c_2 m_2) = \pi_*^{-1}(I_1, m_1) + \pi_*^{-1}(I_2, m_2)$.

Now we introduce the main object in the order reduction process.

Def 1.5 ① Let (X, I, m, E) be the object (resp. (X, I, E))

(1) X is sm var $/k$ char $=0$

(2) $0 \neq (I, m) \subset \mathcal{D}_X$ marked ideal (resp. $0 \neq I \subset \mathcal{O}_X$ ideal)

(3) ☆ $E := (E^1, \cdots, E^s)$ an ordered set of smooth divisors, s.t. $\sum E^i$ is snc, E^i can be zero divisor.

has no order, just a snc div. (X, I, E)

② And a smooth blow-up of (X, I, m, E) (resp. (X, I, E)) is a sm blow-up $\pi: Bl_Z X \to X$ such that

(1) Z is snc with $\bigcup_{i=1}^{s} E^i$ ☆ (2) $Z \subseteq \operatorname{supp}(I, m)$

③ The bir transform of (X, I, m, E) (resp. (X, I, E)) is

$$\pi_*^{-1}(X, I, m, E) = (Bl_Z X, \pi_*^{-1}(I, m), \pi_{tot}^{-1} E)$$

here $\Rightarrow \pi_{tot}^{-1} E = (\pi_*^{-1} E^1, \pi_*^{-1} E^2, \cdots, \pi_*^{-1} E^s, E^{s+1} = Excep(\pi))$

$$\pi_*^{-1}(X, I, E) = (Bl_Z X, \pi_*^{-1} I, \pi_{tot}^{-1} E).$$

Remark: In the principlization we introduced triple (X, I, E) with E just snc div. and the "transform" of (X, I, E) on $Bl_Z X$ is $(Bl_Z X, \pi^{-1} I \cdot \mathcal{O}_{Bl_Z X}, \pi_*^{-1} E)$.

Def 1.6. (Sequence of blow-ups for (X, I, m, E) and (X, I, E)).

① A smooth blow-up seq of (X, I, m, E) is $(E = (E^1, E^2, \cdots, E^s))$

$\boxed{BO}$: $\pi: (X_r, I_r, m_r, E_r) \to (X_{r-1}, I_{r-1}, m_{r-1}, E_{r-1}) \to \cdots \to (X_0, I_0, m_0, E_0)$

s.t. (1) Each blow-up is smooth with center Z_i snc with $\bigcup_{i=1}^{s} E^i$

(2) $(I_{i+1}, m) = \pi_*^{-1}(I_i, m)$, $E_{i+1} = \pi_{tot}^{-1} E_i$

☆(3) $Z_i \subseteq \operatorname{supp}(I_i, m_i)$.

② A smooth blow-up seq of (X, I, E) of $\boxed{\text{order } m}$ is

$\boxed{B^m}$: $\pi: (X_r, I_r, E_r) \to \cdots \to (X_0, I_0, E_0)$

$(1)^* = (1)$

$(2)^*$: $I_{i+1} = \pi_*^{-1} I_i$ $E_{i+1} = \pi_{tot}^{-1} E_i$

$(3)^*$: ☆ $\operatorname{Ord}_{\eta_{Z_i}} I_i = m$ $\forall 0 \leq i \leq r-1$.

Rem: In both definitions, empty blow-ups are not allowed, but trivial blow-ups are allowed.

Prop 1.7. Let $h: Y \to X$ be a smooth morphism, then for any $BO(X, I, m, E)$ (resp. $B^m(X, I, E)$), $h^* BO(X, I, m, E)$ (resp. $h^* B^m(X, I, E)$) is well defined.

Proof:

Fact: $\forall y \in Y$, $x = h(y)$, we have

$$\mathrm{ord}_y\, h^{-1} I \cdot O_Y = \mathrm{ord}_x I. \text{ — Def 1.3 Rem ⑦.}$$

$$\tilde{\pi}_i : Y_{i+1} \to Y_i$$

$$\boxed{\begin{array}{l} J_r = h^{-1} I_r \cdot O_{Y_r} \\ \vdots \\ J_2 = h^{-1} I_2 O_{Y_2} \\ J_1 = h^{-1} I_1 \cdot O_{Y_1} \\ J = h^{-1} I \cdot O_Y \end{array}}
\qquad
\begin{array}{ccccc}
Y_0 \times X_r & \xrightarrow{h_r} & X_r & & I_r \\
\downarrow & & \downarrow & & \\
\vdots & & \vdots & & \\
Y_0 \times X_2 & \xrightarrow{h_2} & X_2 & & I_2 \\
\downarrow & & \downarrow \pi_1 & & \\
Y_0 \times X_1 & \xrightarrow{h_1} & X_1 & & I_1 \\
\downarrow & & \downarrow \pi_0 & & \\
Y_0 & \xrightarrow{h} & X_0 & & I_0
\end{array}$$

We need to check: $(J_{i+1}, m) = \tilde{\pi}_{i*}^{-1}(J_i, m)$.

$$\Leftrightarrow \left(h_{i+1}^{-1} I_{i+1} \cdot O_{Y_{i+1}}, m \right) = \tilde{\pi}_{i*}^{-1} \left(h_i^{-1} I_i \cdot O_{Y_i}, m \right)$$

$$\Leftrightarrow \left(h_{i+1}^{-1} I_{i+1} \cdot O_{Y_{i+1}}, m \right) = \tilde{\pi}_i^{-1} \left(h_i^{-1} I_i \cdot O_{Y_i}, m \right) \cdot O_{Y_{i+1}}\left(m\, h_{i+1}^{-1}(E_{i+1}) \right).$$

E_{i+1} is the excep div of $\pi_i : X_{i+1} \to X_i$.

$$\Big\Vert$$

$$h_{i+1}^{-1} \left(\pi_i^{-1} I_i \cdot O_{X_{i+1}} \cdot O_{X_i}(m E_i) \right) \cdot O_{Y_{i+1}}, m).$$

$$\begin{array}{ccc}
Y_{i+1} & \xrightarrow{h_{i+1}} & X_{i+1} \\
\tilde{\pi}_i \downarrow & \circlearrowleft & \downarrow \pi_i \\
Y_i & \xrightarrow{h_i} & X_i
\end{array}$$

Rem: We say BO (resp B^m) commutes with sm mor if

may not be surjective. ordered $=$.

$h^* BO(X, I, m, E)$ is an extension of $BO(Y, h^{-1} I \cdot O_Y, m, h^{-1} E)$ (resp. B^m). ☆(Just as blow-up sequence)☆

Rem: for closed embeddings, as we mentioned before $j: x \in X \hookrightarrow A$, it can happen that

$$\mathrm{ord}_x\, j_* I \cdot O_A = 1 \quad !!!$$

so functoriality resp to closed embedding in general does not make sense.

I_X in O_A contain some smooth element.

But, for $(I, 1)$, functoriality resp to closed embedding still make sense.

Now, we introduce two order reduction theorem

Ord I (ord reduction for ideals)

$\boxed{\text{For every } m \in \mathbb{N}}$, there is a smooth blow-up sequence functor
Necessary.
$\mathcal{B}^m$ $\boxed{\text{of order } m}$ defined on $(X, I, E := (E^1, \cdots, E^s))$, $\boxed{\text{maxord } I \le m}$

$$\mathcal{B}^m: (X_r, I_r, E_r) \to \cdots \to (X_1, I_1, E_1) \to (X_0, I_0, E_0)$$
$$\qquad\qquad\qquad\qquad\qquad\qquad \cup \qquad\qquad\qquad \cup$$
$$\qquad\qquad\qquad\qquad\qquad\qquad Z_1 \qquad\qquad\qquad Z_0$$

i.e.

(each center Z_i snc with E, $\mathrm{Ord}_{\eta_{Z_i}} I_i = m$)

(1) max-ord $I_r < m$

(2) $\mathcal{B}^m$ commutes with smooth morphism and change of fields.

Ord II (Order reduction theorem for marked ideals)

$\boxed{\text{For every } m \in \mathbb{N}}$, there is a smooth blow-up seq functor BO defined on (X, I, m, E) such that

E: ordered set of sm div.

$$BO: \Pi: (X_r, I_r, m, E_r) \to \cdots \to (X, I, m, E)$$

i.e.

$(Z_i \subseteq \mathrm{Supp}(I_i, m)$, Z_i snc with $E)$ $\to$ $\mathrm{ord}_{\eta_{Z_i}} I_i \ge m$ $(m = \text{maxord})$ $\Downarrow$ $\mathrm{ord}_{\eta_{Z_i}} I_i = m$.

(1) $\mathrm{Supp}(I_r, m) = \emptyset$

(2) commutes with sm mor and change of fields.

Remark: If $m = \text{max-ord } I$, $\mathcal{B}^m(X, I, E) = BO(X, I, m, E)$; ☆

In this case, $\mathrm{ord}_{\eta_{Z_i}} I_i = m$.

$\pi \dashv \mathcal{I}$. $\quad R \xrightarrow{\pi} R/A$. $\quad \mathcal{I}$

$$j^{\#}: \mathcal{O}_X \to j_* \mathcal{O}_S.$$

$$j_* \mathcal{I}_S \cdot \mathcal{O}_X = j^{\# -1}(j_* \mathcal{I}_S).$$

$\boxed{\text{OCE}}$: $j: S \longrightarrow X$ closed embedding, $\mathcal{I}_S$

$$BO(X, j_* \mathcal{I}_S \cdot \mathcal{O}_X, 1, \emptyset) = j_* BO(S, \mathcal{I}_S, 1, \emptyset)$$

The main inductive steps of the proof is

$$\text{Ord II in dim} \leq n-1$$

$\Downarrow T_1 \quad \to \boxed{\text{use restriction.}}$ ☆.

→ the induction has nothing to do with "m". ☆

$$\text{Ord I in dim } n$$

$$\Downarrow T_2$$

$$\text{Ord II in dim } n.$$

$\boxed{\text{We only use } m = 1}$

By Ord II + OCE $\boxed{\overset{T_3}{\Longrightarrow}}$ P III $\Rightarrow$ Main Goal.

Now we prove T_3.

Proof of T_3: We start from a triple (X, I, E) ← just snc divisor.

Step 1: Write $E = \bigsqcup_{i=1}^{k} D_i$, set $\tilde{E} = (D_1, D_2, \cdots, D_k)$ D_i sm div.

For any point $x \in X$, set $S_{X,E}(x) = \{$number of div in $\{D_i\}$ passing $x\}$.

$$S(X, E) = \max\{S_{X,E}(x) \mid x \in X\}.$$

Set $\underline{H^{S(X,E)}} = \bigsqcup_{\substack{A \in \{1, \cdots, k\} \\ |A| = S(X,E)}} \bigcap_{i \in A} D_i$ we blow up $H^{S(X,E)}$.

Here $H^{S(X,E)}$ is a smooth center.

$\Pi_0: X_1 \to X_0 = X$ is a sm blow-up of center $H^{S(X,E)}$

Consider the corresponding $S_{X_1, \Pi_{0*}^{-1} E}(x_1)$ $\quad S(X_1, \Pi_{0*}^{-1} E)$.

$$\downarrow$$

$$(\Pi_*^{-1} D_1, \cdots, \Pi_*^{-1} D_k).$$

$$S(X_1, \Pi_{0*}^{-1} E) < S(X, E).$$

Repeat this procedure, we construct $\widetilde{B}: X_r \to X_{r-1} \to \cdots \to X_0 = X$ s.t. $(\Pi_i : X_i \to X)$

$$0 = S(X_r, \Pi_{r*}^{-1} E) < S(X_{r-1}, \Pi_{r-1*}^{-1} E) < \cdots < S(X, E)$$

$\Rightarrow$ $\Pi_{r*}^{-1} E$ is a disjoint union of irr sm components,

We now show $\widetilde{B}(X, I, E)$ is functorial resp to sm morp.
$\forall\ h: Y \to X$ $y \in Y$, $x = h(y)$, we have $S_{X,E}(x) = S_{Y, h^{-1}E}(y)$

thus $S(X,E) = S(Y, h^{-1}E)$. Since each time we blow up the maximal locus of $S_{X,E}$ $(S_{Y,h^{-1}E})$ $\Rightarrow$ $H^{S(Y,h^{-1}E)} = h^{-1} H^{S(X,E)}$

thus the functoriality resp to sm morp follows.

Rem: This idea is used in [W£005], where we use a function to control the blow up process, and this function is inv resp to sm mor.

Step 2: Now consider $(X_r, \underbrace{\Pi_r^{-1} I \cdot \mathcal{O}_{X_r}}_{J}, (\underbrace{\Pi_{r*}^{-1} E}_{F_0}, F_1, F_2, \cdots, F_r))$

here $\Pi_{r*}^{-1} E$ is smooth.

$m = 1$.

Apply Ord II to $(X_r, J, 1, F)$, we get
$BO(X_r, J, 1, F): \Pi: X_n \to \cdots \to X_r$
s.t. $\operatorname{Supp} \Pi_*^{-1}(J, 1) = \phi$.
$\Rightarrow$ $\Pi_1^{-1} J \cdot \mathcal{O}_{X_n} = \mathcal{O}_{X_n}(\operatorname{Excep} \Pi)$ here $\operatorname{Excep} \Pi$ is snc, Π exceptional

And $X_n \to X_r \to X_0$ gives the principalization.
Sm functoriality follows from Ord II and construction.
functoriality resp to closed embedding follows from OCE.

P III $E = \phi$
in this case, step 1 is trivial.