Resolution of Singularities.

§0. Preliminarlies and Main Goal.

A variety is an integral seperate scheme of finite type over a field k.

Main Goal: Let X be a variety over a field of char zero. Then there exists a canonical desingularization of X, that is a smooth variety $\widetilde{X}$ and a proj bir morphism

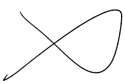
$$res_X : \widetilde{X} \longrightarrow X$$

such that

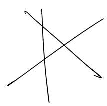
(1) $res_X^{-1}(\mathrm{Sing}\, X)$ is a divisor with simple normal crossings.

(2) res_X is functorial respect to smooth morphisms and field extension.

Remark: $E = \sum E^i$, E^i irr is called snc (on smooth var) if E^i is smooth, and for each closed pt, $\exists$ local cors $z_1, \cdots z_n$ s.t. $E^i = (z_{k_i}^{d_i} = 0)$, $E = (\prod z_{k_i}^{d_i} = 0)$.



normal crossing



snc.

Remark: Res "functor", associate each object $X \in Var$ an $\widetilde{X} \in Var$ and a bir proj $res_X : \widetilde{X} \to X$.

we say it is functorial resp to sm mor if $\forall$ $h: Y \to X$ smooth mor, $\exists$ $\widetilde{h}: \widetilde{Y} \to \widetilde{X}$ s.t.

the diagram
$$\begin{tikzcd} \widetilde{Y} \arrow[r, "\widetilde{h}"] \arrow[d] & \widetilde{X} \arrow[d, "res_X"] \\ Y \arrow[r, "h"] & X \end{tikzcd}$$
is a fiber product.

Prop 1: Let Res be a resolution "functor" that is functorial resp to sm mor, then $\forall$ var X, $res_X : \widetilde{X} \to X$ is an isomorphism over $X \setminus Sing(X)$.

Proof: Let $x \in X$ be a smooth point.

$$\begin{tikzcd} X \ni x \arrow[d] & U \arrow[l, hook', "i"', "\text{open}"] \arrow[r, "\pi"] & \mathbb{A}_k^{\dim X} \arrow[dll] \\ k & & \end{tikzcd} \qquad \pi \text{ is étale}$$

Claim 1*: G is an algebraic group $/k$ char 0, then $res_G : \widetilde{G} \to G$ is an isomorphism.

Assuming claim 1*:

since
$$\begin{tikzcd} U \arrow[r, "\pi"] & \mathbb{A}_k^{\dim X} \\ \widetilde{U} \arrow[u, "res_U"] \arrow[r, "\widetilde{\pi}"'] \arrow[ur, phantom, "\circlearrowright"] & \mathbb{A}_k^{\dim X} \arrow[u, "res = id"'] \end{tikzcd}$$
is a fiber product

$\Rightarrow$ res_U is an isomorphism.

Now,
$$\begin{tikzcd} x \in X & U \arrow[l, hook'] \\ \widetilde{X} \arrow[u, "res_X"] \arrow[ur, phantom, "\circlearrowright"] & U \arrow[l] \arrow[u, "id"'] \end{tikzcd} \Rightarrow res_X \big|_U \text{ is an iso}$$

$\square$

Proof of claim 1*: suppose $res_G : \widetilde{G} \to G \ni g_1$ is not iso on g_1.

$$\begin{tikzcd} \widetilde{G} \arrow[r, "\widetilde{\phi}"] \arrow[d, "res_G"'] & \widetilde{G} \arrow[d, "res_G"] \\ G \arrow[r] & G \\ g_2 \arrow[r, "g_1 \cdot g_2^{-1}", "\underset{\phi}{\shortparallel}"'] & g_1 \end{tikzcd}$$

here g_2 is a general pt s.t. res_G is iso,

$$\Rightarrow 1 \le \dim \widetilde{\phi}^{-1} res_G^{-1}(g_1) = \dim res_G^{-1} \phi^{-1}(g_1) = \dim res_G^{-1}(g_2) = 0$$

$\Rightarrow \Leftarrow$

$\square$

Prop 2: Let Res be a resolution "functor" as above.
Let $X \in Var/k$ char $= 0$, G is an algebraic group acting on X, then the G-action lifts to a G action on $\widetilde{X}$ s.t.
$res_X : \widetilde{X} \to X$ is G-equivariant. i.e.
$res_X(g(\widetilde{x})) = g\, res_X(\widetilde{x})$.

Proof: Note that the G-action is a smooth morphism.

i.e. $G \times X \xrightarrow{\phi_G} X$ is smooth.

① Consider $\pi_1 : G \times X \longrightarrow X$ projection.

$$\begin{tikzcd} \widetilde{(G\times X)} \arrow[r, "\widetilde{\pi}_1"] \arrow[d, "res"'] \arrow[dr, phantom, "\circlearrowleft"] & \widetilde{X} \arrow[d] \\ G\times X \arrow[r, "\pi_1"'] & X \end{tikzcd}$$

is fiber product $\Rightarrow \widetilde{(G\times X)} \cong G\times\widetilde{X}$.

Consider $\phi_G : G\times X \to X$.

$$\begin{tikzcd} G\times\widetilde{X} \arrow[r, "\widetilde{\phi}_G"] \arrow[d] & \widetilde{X} \arrow[d] \\ G\times X \arrow[r, "\phi_G"'] & X \end{tikzcd}$$

we now show that $\widetilde{\phi}_G$ gives a group action on $\widetilde{X}$.

②.

$$\begin{tikzcd} G\times G\times X \arrow[r, "Id\times\phi_G"] \arrow[d, "m_G\times Id"'] \arrow[dr, phantom, "\circlearrowleft"] & G\times X \arrow[d, "\phi_G"] \\ G\times X \arrow[r, "\phi_G"'] & X \end{tikzcd}$$

$$\begin{tikzcd} G\times G\times\widetilde{X} \arrow[r] \arrow[d] & G\times\widetilde{X} \arrow[d] \\ G\times\widetilde{X} \arrow[r] & \widetilde{X} \end{tikzcd}$$

We know, the diagram commutes over $X \setminus Sing X$.

$$\begin{tikzcd} g_1\times g_2\times\widetilde{X}\ni\widetilde{x} \arrow[r] \arrow[d] \arrow[drr, bend left, "\phi_1"] \arrow[dr, bend right, "\phi_2"'] & g_1\times g_2\widetilde{X} \arrow[d] \\ g_1g_2\times\widetilde{X} \arrow[r] & \widetilde{X} \end{tikzcd}$$

$\forall \widetilde{x} \in \widetilde{X}$, $\exists$ a general smooth curve not in $res^{-1}(Reg X)$. denoted as $\widetilde{C} \ni \widetilde{x}$,

$\phi_1(C) = \phi_2(C) / (X \setminus Sing X) \xRightarrow{\text{by valuative criteria}} \phi_1(C) = \phi_2(C)$

$\Rightarrow \phi_1(\widetilde{x}) = \phi_2(\widetilde{x})$

$\square$.

One of the key idea is, instead of considering the resolution problem, we consider the "Principalization" problem.

Easy version, weak.

PI: Let X be a smooth variety over $/k$ char $=0$, $I \subset \mathcal{O}_X$ non-zero ideal sheaf.
Then $\exists\ f: \tilde{X} \to X$ proj bir, $\tilde{X}$ sm, such that

$$f^{-1}I \cdot \mathcal{O}_{\tilde{X}} \subset \mathcal{O}_{\tilde{X}} \quad \text{is an invertible ideal sheaf.}$$

Rem: $f^{-1}I \cdot \mathcal{O}_{\tilde{X}}$ is the image of f^*I under $f^*\mathcal{O}_X \to \mathcal{O}_{\tilde{X}}$.

Cor I: (Elimination of indeterminacies).
Let X be a smooth variety $/k$ char $=0$, $g: X \dashrightarrow \mathbb{P}$ a rational map to some proj space. Then $\exists$ sm var $\tilde{X}$ and proj bir mor $f: \tilde{X} \to X$ such that $g\circ f: \tilde{X} \to \mathbb{P}$ is a morphism.

$$\begin{tikzcd} \tilde{X} \arrow[d, "f"'] \arrow[dr, "g\circ f"] & \\ X \arrow[r, dashed, "g"'] & \mathbb{P} \end{tikzcd}$$

Proof: Since $\mathbb{P}$ is projective, $\exists\ Z \subseteq X$ with codim ≥ 2 s.t $g: X\setminus Z \to \mathbb{P}$ is a morphism. (Valuative criterion for properness)
By algebraic Hartogs thm, $g^*\mathcal{O}(1)|_{X\setminus Z}$ extends uniquely to a linebundle on X, denoted as L.
Let $J \subset L$ be a subsheaf generated by $g^*H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1))$, $I = J \otimes L^{-1} \subset \mathcal{O}_X$
Take $f: \tilde{X} \to X$ s.t. $f^{-1}I \cdot \mathcal{O}_X$ is invertible ideal sheaf.

Since $f^*I = f^*J \otimes (f^*L)^{-1} \Rightarrow f^*J = f^*I \otimes f^*L$.

$\tau: f^*I \hookrightarrow \mathcal{O}_{\tilde{X}}$ $f^{-1}I\cdot\mathcal{O}_{\tilde{X}}$ that defines $f^{-1}I\cdot\mathcal{O}_{\tilde{X}}$

$\tau \otimes f^*L$: $f^*I \otimes f^*L = f^*J \twoheadrightarrow f^{-1}I\cdot\mathcal{O}_{\tilde{X}} \otimes f^*L$. invertible.

$\Rightarrow \operatorname{Im}(f^*J)$ is a subsheaf of L' generated by $(g\circ f)^* H(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)) \Rightarrow$ No base locus on $\tilde{X} \Rightarrow$ defines a mor.

Before we state a stronger version of "principalization", let's recall some notations.

Smooth blow-up : $Z \overset{\text{closed}}{\subset} X$, $\pi : Bl_Z : \tilde{X} \to X$

We say π is a smooth blow-up if X, Z are both smooth, Z sm in X.

We say π is trivial if Z is Cartier, π is iso.

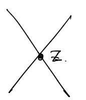
We say π is empty if $Z = \phi$, π is iso.

snc — center : E is a snc divisor, $E = \sum E_i$, $Z \subset X$

We say Z is snc with E if $\exists$ local cor system $\{z_1, \dots, z_n\}$

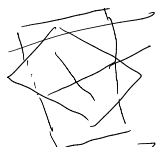
s.t. $Z = (z_{j_1} = \cdots = z_{j_s} = 0)$ $\quad E_i = (z_{c(i)} = 0)$

If $Z \not\subseteq \operatorname{Supp} E$, then $E|_Z$ is snc.



z.

E



L snc.

$$\begin{array}{ccccccccc} & & Z_{n-1} & & & & Z_1 & & Z_0 \\ & & \cap & & & & \cap & & \cap \\ X_n & \longrightarrow & X_{n-1} & \longrightarrow & \cdots & \longrightarrow & X_1 & \longrightarrow & X_0 = X \end{array}$$

Let $\pi :$ $X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 = X$ be a seq of blow-ups of sm variety X, E snc div on X, Z_i is snc with E if

$Z_i \subset X_i$ is snc with $\pi_{i*}^{-1} E + \sum_{j<i} \pi_{ij}^{-1}(Z_j)$ (snc).

Here. $\pi_i : X_i \to X$ $\quad \pi_{ij} : X_i \to X_j$.

When $E = \phi$, Z_i is snc with exceptional set.

Now we state a stronger version.

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PII : Let X be a sm variety $/k$ char $=0$, $I \subset \mathcal{O}_X$ a nonzero ideal sheaf, E snc div on X, $\exists$ seq of blow-ups (sm)

$$\pi : \tilde{X} = X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 = X$$

whose center has snc with E, such that (center smooth)

(1) $\pi^{-1} I \cdot \mathcal{O}_{\tilde{X}}$ is the ideal sheaf of a snc div, and (snc with excep).

(2) π is functorial resp. to sm morphism.

Rem : (2) gaurantees that $\pi|_{X \setminus \mathrm{Supp}\, I}$ is an iso.

Cor II : (Non functorial "weak" "Embedded" Resolution of Sing)

Let Y be a closed subvariety of a sm variety X/k char 0. Then there is a bir proj mor $\pi : \tilde{X} \to X$ such that π is iso near η_Y i.e. $\pi|_Y : \tilde{Y} \to Y$ proj bir, and $\tilde{Y}$ has snc with $\sqcup$ excep divs on $\tilde{X}$.

$\downarrow$ smooth. $\longrightarrow (\mathbb{A}^n, \mathcal{O}_{\mathbb{A}^n})$

(Not sure $\pi|_Y$ is iso over $Y \setminus \mathrm{Sing}\, Y$, $(\pi|_Y)^{-1}(\mathrm{Sing}\, Y)$ snc on $\tilde{Y}$).

Proof : (of CII assuming PII)

Let I_Y be the ideal sheaf of $Y \subseteq X$. Let $\pi_r : X_r \to X_{r-1} \to \cdots \to X_1 \to X_0 = X$ be a seq of sm blow-ups whose centers are snc with ϕ as in PII such that $\pi_r^{-1} I_Y \cdot \mathcal{O}_{X_r}$ is principal.

① If π_r is an iso over $\eta_Y \subseteq X$, since $\pi^{-1} I_Y \cdot \mathcal{O}_{X_r}$ is an snc div, we can find an irr comp of $\mathrm{Supp}\, \pi^{-1} I_Y \cdot \mathcal{O}_{X_r}$, denoted as $\tilde{Y}$ and $\tilde{Y} \to Y$ is bir, $\tilde{Y}$ is smooth.

② If π_r is not an iso ~~over~~ over η_Y. $\exists j$ s.t. $\eta_Y \subseteq Z_j$ and $\eta_Y \nsubseteq Z_i$ for $i<j$. Since π is iso over $X \setminus Y$,

and $\pi_j(Z_j) \subseteq \mathrm{Supp}(I_Y) = Y \Rightarrow \exists$ irr comp of Z_j, denoted as $\tilde{Y}$ s.t. $\tilde{Y} \to Y$ bir.

Remark: ① In PII, only $\pi: \tilde{X} \to X$ is "functorial" resp to sm mor, but we
we choose middle blow up $\pi_j : X_j \to X$, "func" fails.

Let's introduce some notations and definitions.

Def 1: (Blow-up sequence) Let X be a var, a blow-up seq of length r starting with X is a chain of morphisms

$$\begin{tikzcd} \mathbb{B}: \quad \pi: X_r \arrow[r, "\pi_{r-1}"] & X_{r-1} \arrow[r] & \cdots \arrow[r, "\pi_1"] & X_1 \arrow[r, "\pi_0"] & X_0 = X \\ & Z_{r-1} \arrow[u, phantom, "\cup"] & & Z_1 \arrow[u, phantom, "\cup"] & Z_0 \arrow[u, phantom, "\cup"] & Y \arrow[ul] \end{tikzcd}$$

where each π_i is a blow up of X_i with center Z_i.

$\pi_{ij}: X_i \to X_j \qquad \pi_i : X_i \to X$

We say a blow-up seq is smooth if X_i, Z_i are all smooth

We allow trivial and empty blow-ups in the seq X.

Def 2 (Pull back blow-up seq by sm morphism)

$\mathbb{B}$ be a blow-up seq as above. $h: Y \to X$ smooth morphism.

$$\begin{tikzcd} h^*\mathbb{B} \quad h^*\pi: X_r \times_X Y \arrow[r] & X_{r-1} \times_X Y \arrow[r] & \cdots \arrow[r] & X_1 \times_X Y \arrow[r] & X_0 \times_X Y = Y \\ & Z_{r-1} \times_X Y \arrow[u, phantom, "\cup"] & & Z_1 \times_X Y \arrow[u, phantom, "\cup"] & Z_0 \times_X Y \arrow[u, phantom, "\cup"] \end{tikzcd}$$

$h^*\mathbb{B}$ is called the pull back of $\mathbb{B}$ by h.

Rem: ① $h^*\mathbb{B}$ defines a blow up seq. i.e.

$$\begin{tikzcd} Bl_Z X \times_X Y \arrow[r] \arrow[d, "\pi_Y"'] & Bl_Z X \arrow[d, "\pi_X"] \\ Y \arrow[r, "h"] & X \end{tikzcd}$$

π_Y is a blow-up of $h^{-1} Z$.

In general, h not smooth, $h^*\mathbb{B}$, see examp

$$\begin{tikzcd} S \arrow[r] \arrow[d] & Bl_0 \mathbb{P}^2 \arrow[d] \\ Bl_0 \mathbb{P}^2 \arrow[r, "h"] & \mathbb{P}^2 \end{tikzcd}$$

$S \to Bl_0 \mathbb{P}^2$ obviously not a blow up $\quad S = Bl_0 \mathbb{P}^2 \times_{\mathbb{P}^2} Bl_0 \mathbb{P}^2$.

② $\mathbb{B}$ is smooth, then $h^*\mathbb{B}$ is smooth blow-up

③ h is not surjective, $h^*\mathbb{B}$ may contain extra empty blow-up

Def 3 (Restriction to closed subvariety)

Let $\mathbb{B}$ as above, $j: S \to X$ is a closed emb. def

$$\mathbb{B}|_S = \quad \begin{array}{ccccccc} S_r \to & S_{r-1} & \to \cdots \to & S_1 & \to & S_0 = S. \\ & \cup & & \cup & & \cup \\ & Z_{r-1} \cap S_{r-1} & & Z_1 \cap S_1 & & Z_0 \cap S_0 \end{array}$$

here we need $\eta_S \notin Z_j$, (In fact we require all Z_i has image strictly contained in S in application)

Def 4 (Push forward resp to closed embedding)

$j: S \to X$ closed embedding.

$$\mathcal{B}_S: \pi: \begin{array}{ccccccc} S_r \to & S_{r-1} & \to \cdots \to & S_1 & \to & S_0 = S \\ & \cup & & \cup & & \cup \\ & Z^S_{r-1} & & Z^S_1 & & Z^S_0 \end{array} \quad \text{blow-up seq for } S.$$

define

$j_*\mathcal{B}_S$ as $j_*\pi$: $$\begin{array}{ccccccc} X_r \to & X_{r-1} \to \cdots \to & X_2 & \to & X_1 & \to & X_0 = X \\ & \cup & \cup & & \uparrow & & \cup \\ & Z^S_{r-1} & Z^S_2 & & S_1 & & Z^S_0 \\ & & & & \cup & & \\ & & & & Z^S_1 & & \end{array}$$

Remark: if $\mathbb{B}(S)$ is smooth, then $j_*\mathcal{B}(S)$ is smooth.

Now we consider a triple (X, I, E), where X is smooth var, I ideal sheaf $\subseteq \mathcal{O}_X$, div $E \subseteq X$

$\mathbb{B}(X, I, E)$ $\quad \pi: X_r \to X_{r-1} \to \cdots \to X_1 \to X_0 = X$

$I_i = \pi_i^{-1} I \cdot \mathcal{O}_{X_i} \quad E_i = \pi_{i*}^{-1} E$ on X_i

$$\begin{array}{ccc} X_i \times_X Y & h^*(I_i) = h_i^{-1} I_i \cdot O_{X_i \times_X Y} & E_i \times_X Y \\ \downarrow & & \downarrow \\ \underline{X_i} & I_i & E_i \end{array}$$

We extend push forward, pull back for triple.

① $h: Y \to X$ smooth, $\boxed{h^* B(X, I, E)}$: as $h^*(I_i) = h_i^{-1} I_i \cdot O_{X_i \times_X Y}$, $h^*(E_i) = E_i \times_X Y$

② $j: S \to X$ is a closed embedding. $j_* B_S(X, I_S, E_S)$

$E_S \subset S \subset X$ $\quad E_{S_i} \subset S_i \subset X_i$ natural.

def: $j_* I_S \cdot O_X = (j^\#)^{-1}(j_* I_S)$ $\quad j^\#: O_X \to j_* O_S$ (with $j_* I_S \to j_* O_S$)

$O_X / j_* I_S \cdot O_X = j_*(O_S / I_S)$

locally: $j: \operatorname{Spec} R/A \to \operatorname{Spec} R$. $\quad \varphi: R \to R/A$, $\bar{I}$

$j_* \bar{I} \cdot O_{\operatorname{Spec} R} = \varphi^{-1}(\bar{I})$

define $j_*(I_{S_i}) = j_{i*} I_{S_i} \cdot O_{X_i}$ $\quad j_i: S_i \to X_i$

Def 5: (Functorial Package)

① $B(X, I, E)$ commute with smooth mor $h: Y \to X$ if

$\underline{h^* B(X, I, E)}$ is an extension $\underline{B(Y, h^{-1} I \cdot O_Y, h^{-1} E)}$.

extension: $h^* B$ is $B(Y, h^{-1} I \cdot O_Y, h^{-1} E)$ by adding some empty blow-ups.

② $B(X, I, E)$ commutes with closed embedding if

$I_S \ S \xrightarrow{j} X \ E$ $\quad B(X, j_* I_S \cdot O_X, E) = j_* B(S, I_S, E|_S)$

$\Rightarrow j^*(X, j_* I_S \cdot O_X, E) = B(S, I_S, E|_S)$

Now we state the final principalization thm.

P III : For any triple $(X, I, \boxed{E})$ where X is sm var / k char = 0, E snc div on X, $I \subset \mathcal{O}_X$ ideal sheaf, then there exist a smooth blow-up seq functor $\mathcal{B}(X, I, E)$, such that all centers of blow-ups are snc with E, and

(1) $\pi^{-1} I \cdot \mathcal{O}_X$ is an ideal sheaf of snc div,
(2) $\mathcal{B}$ commutes with smooth morphisms,
(3) $\mathcal{B}$ commutes closed embeddings whenever $\boxed{E = \phi}$
(4) $\mathcal{B}$ commutes with field extensions (seperable)

C III : (Functorial Strong "Embedded" Resolution of Sing)
Let Y be a subvar of sm var X/k char 0. Then there exists a seq of smooth blow ups $\pi : \widetilde{X} = X_r \to \cdots \to X_1 \to X_0 = X$ with centers snc with ϕ, such that

(1) $\pi|_Y : \widetilde{Y} \to Y$ is a proj bir with $\widetilde{Y}$ main comp of $\pi^{-1}(Y)$, here π is iso near η_Y.
(2) $\pi|_Y$ is iso over $(Y \setminus \operatorname{Sing} Y)$, $\pi^{-1}(\operatorname{Sing} Y)$ snc on $\widetilde{Y}$.
(3) π commutes with sm morphisms
(4) π commutes with closed embeddings.
(5) π commutes with field extensions.

Rem: The above resolution depends on how we embed varieties, and can only resolve sings of varieties that can be embedded into sm varieties.
Problem 1: Not every var has an embedding to sm var.
Problem 2: To remove the influence of embedding, we need to glue resolutions, but the glued mor may not be projective.
Problem 3: To ensure canonicity, can we find "canonical embedding"?

PIII ⇒ CIII: As in the proof of PII ⇒ CII, we have a seq of smooth blow-ups $\Pi: \widetilde{X} = X_r \to X_{r-1} \to \cdots \to X_1 \to X_0 = X$ which appears as a subseq of the "principalization blow-up seq" of (X, I_Y, ϕ). Such that Π is an iso near η_Y, and Π induce a resolution $\Pi|_Y: \widetilde{Y} \to Y$.

(3)-(5) follows directly from PIII (2)-(3).

Now, for (2) $\forall y \in Y \subset X$ smooth point of Y, $\exists\, y \in U \subset X$ s.t.

$$(\mathbb{A}^n, I_0 = \langle x_1, x_2, \dots, x_k \rangle) \xleftarrow[\text{étale}]{\pi} (U, I_Y|_U)$$

$$I_Y|_U = \pi^{-1} I_0 \cdot \mathcal{O}_U.$$

⇒ take $y' \in Y \subset X$ s.t. $\Pi|_Y$ is an iso near $y' \in Y$

Functoriality
$$\begin{tikzcd} & (U, I_Y|_U) \arrow[dl, "\pi"'] \\ (\mathbb{A}^n, I_0) & \\ & (U', I_Y|_{U'}) \arrow[ul, "\pi'"] \end{tikzcd}$$
⇒ $\Pi|_Y$ is iso near $y \in Y$.

So $\Pi|_Y$ is iso $/ (Y \setminus \mathrm{Sing}\, Y)$.

$\Pi^{-1}(\mathrm{Sing}\, Y)$ is snc, this follows from all blow up center is snc with ϕ.

L1 (Gluing Lemma)

Let $\mathbb{B}$ be a blow-up seq "functor" defined for affine varieties over k of char 0, such that commutes with smooth morphism, then $\mathbb{B}$ has a unique extension to all varieties.

Proof: Let X be a variety $/k$, $\{U_i\}$ an affine cover

For each i, $\mathbb{B}$ assigns a center Z_{i0} on U_i s.t. the first blow up is $Bl_{Z_{i0}} U_i \longrightarrow U_i$ for $\mathbb{B}(U_i)$.

Consider $\phi_i^\circ: U_{ij} \to U_i$ $\quad \phi_j^\circ: U_{ij} \to U_j \quad (U_{ij} = U_i \cap U_j)$

$$\phi_i^{\circ *} \mathbb{B}(U_i) = \mathbb{B}(U_{ij}) = \phi_j^{\circ *} \mathbb{B}(U_j)$$

⇒ $Z_{i0}|_{U_{ij}} = Z_{j0}|_{U_{ij}} = Z_{ij0}$ ⇒ we can glue all Z_{i0} naturally to Z_0 on X, and get a "canonical" glued blow-up $Bl_{Z_0} X \to X$.

→ this gaurantees the projectivity.

This process does not depend on choice of $\{U_i\}$, since for other cover $\{V_i\}$, then we can repeat above process to $\{U_i\} \cup \{V_i\}$, this solves the problem.
Repeat the process, we can construct a seq of blow-up functor for X, and functoriality for sm morphism and field extension follows from the construction.

$\square$.

Rem: For any $h: Y \to X$, we can write it as $h|_{U_i}: h^{-1}(U_i) \to U_i$, and apply gluing argu.

L2 (Local canonicity of embedding)
Let X be an affine variety X and $i_1 X \hookrightarrow A^n$, $i_2 X \hookrightarrow A^m$ two closed embeddings then we have a further embedding

$$i_1': X \xrightarrow{i_1} A^n \longrightarrow A^{n+m} \quad \text{and} \quad i_2': X \xrightarrow{i_2} A^m \longrightarrow A^{n+m}$$
$$x \to i_1(x) \to (i_1(x), 0) \qquad x \to i_2(x) \to (0, i_2(x))$$

such that i_1' and i_2' are equivalent under a (nonlinear) automorphism of A^{n+m}.

Proof: We extend i_1 to $j_1: A^m \to A^n$; the extension is not unique.
i_2 to $j_2: A^n \to A^m$.

$$\begin{tikzcd} & X \arrow[dl, "i_1"'] \arrow[dr, "i_2"] & \\ A^n \arrow[rr, shift left, "j_2"] & & A^m \arrow[ll, shift left, "j_1"] \end{tikzcd} \Rightarrow \begin{tikzcd} R_x & k[y^m] \arrow[l, two heads, "\varphi"'] \\ k[x^n] \arrow[u, two heads, "\phi"] \arrow[ur, "j^\#"'] & \end{tikzcd}$$

: $\forall x_i$, take $y_i \in \varphi^{-1}\phi(x_i)$ def $j^\#: x_i \to y_i$..

Let $\vec{x}$ be cords on A^n, $\vec{y}$ on A^m.

$\varphi_1: (\vec{x}, \vec{y}) \longrightarrow (\vec{x}, \vec{y} + j_2(\vec{x})) \qquad A^{n+m} \to A^{n+m}$

$\varphi_2: (\vec{x}, \vec{y}) \longrightarrow (\vec{x} + j_1(\vec{y}), \vec{y})$.

$$X \xrightarrow{i_1'} A^{n\times m} \xrightarrow{\varphi_1} A^{n\times m}$$
$$\vec{x} \qquad (i_1(x), 0) \qquad (i_1(x), j_2 i_1(x) = i_2(x)).$$
$$X \xrightarrow{i_2'} A^{n\times m} \xrightarrow{\varphi_2} A^{n\times m}$$
$$\vec{x} \qquad (0, i_2(x)) \qquad (i_1(x), i_2(x)).$$

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(In char 0, k is automatically an infinite field).

L3: Let $h: Y \to X$ be a smooth morphism, $y \in Y$ a closed point and $i: X \to A_X$ closed embedding to sm var. Then $\exists$ open set $f(y) \in A_X^\circ \subseteq A_X$, a smooth affine var A_Y° with a smooth morphism

$$h_A: A_Y^\circ \to A_X^\circ$$

set $Y^\circ = h_A^{-1}(X \cap A_X^\circ) \ni y$, it has a closed embedding $Y^\circ \xrightarrow{j} A_Y^\circ$, such that the following diagram commutes and is fiber product.

$$\begin{tikzcd} Y^\circ \arrow[r, "j"] \arrow[d, "h"'] & A_Y^\circ \arrow[d, "h_A"] \\ X^\circ \arrow[r, "i"] & A_X^\circ \end{tikzcd}$$

aff $\to A^N$.

Proof: The problem is local, we assume X, Y, A_X affine, $Y \subset X \times A^N$.
If h is rel of dim d, consider the closed pt $x = h(y)$, by taking general projection $\sigma: A^N_x \to A^{d+1}_x$, we may assume $h^{-1}(x) \to A^{d+1}_x$ is finite mor and is an embedding near $y \in h^{-1}(x)$. (Need k to be infinite field).
Now, shrinking Y and X, we may assume Y is an open subset of a hypersurface $H \subset X \times A^{d+1}$, defined $\sum_I \phi_I Z^I$, ϕ_I reg func on X, Z cor for A^{d+1}.
Now, $X \hookrightarrow A_X$ closed embedding, we extend ϕ_I to Φ_I regular functions on A_X,
set $A_Y = (\sum \Phi_I Z^I = 0) \subset A_X \times A^{d+1} \to A_X$

$\Rightarrow Y \subset A_Y$ and $A_Y \to A_X$ is smooth near $y \in A_Y$.

$\Rightarrow$
$$\begin{tikzcd} y \in Y^\circ \arrow[r, "j"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & A_Y^\circ \arrow[d, "h_A"] \\ X^\circ \arrow[r, "i"] & A_X^\circ \end{tikzcd}$$

Return to our Main Goal: C III $\to$ Main Goal.
By C III we construct $\mathbb{B}(X)$ for affine varieties.
We need to check
① $\mathbb{B}(X)$ is indep of choice of embedding
②. $\mathbb{B}(X)$ functorial respect to smooth morphism
③ $\mathbb{B}(X)$ is functorial resp field ext.

① Follows from L2 :
$$\begin{array}{l} i_1 X \to Y_1 \to A^N \\ \qquad\qquad\qquad \| \to Id. \\ i_2 X \to Y_2 \to A^N \end{array}$$

By CIII (4). we have the uniqueness.

② Let $h: Y \to X$ smooth, by L3, $\forall y \in Y$, we fix embedding $X \to A_X$, $\exists$
$$\begin{tikzcd} Y \arrow[d, "h"] & \supset & Y^\circ \arrow[r, hook] \arrow[d, "h^\circ"] \arrow[dr, phantom, "\circlearrowleft"] & A_Y^\circ \arrow[d, "h_A"] \\ X & \supset & X^\circ \arrow[r, hook] & A_X^\circ \end{tikzcd}$$
that is a fiber product.

Apply CIII (3) to $(A_Y^\circ, I_{Y^\circ}) \to (A_X^\circ, I_{X^\circ})$ we have

$h_A^* \mathbb{B}(A_X^\circ, I_{X^\circ}, \phi)$ is an extension of $\mathbb{B}(A_Y^\circ, I_{Y^\circ}, \phi)$

$\Rightarrow h^* \mathbb{B}(X^\circ) = \mathbb{B}(Y^\circ)$ (as extension)

$\Rightarrow$ by argue as in L1, we have $\mathbb{B}$ is functorial resp sm morp.

③ Follows from CIII (5).

Now, we defined $\mathbb{B}$ for affine vars, by L1, we extend uniquely to a resolution "functor" for all vars / k char k=0.

$\square$

Cor (Log resolution) : Let Y be a closed subscheme in a variety X, then there exists a birational proj morphism $f: \widetilde{X} \to X$ such that $\widetilde{X}$ is smooth and

$f^{-1}(Y) \sqcup Excep$ is a snc div on $\widetilde{X}$

Proof: By "Main Goal", we have

$f_1 : \widetilde{X}_1 \to X$ s.t. $\widetilde{X}_1$ is smooth.

Now consider $(\widetilde{X}_1, f_1^{-1} I_Y \cdot O_{\widetilde{X}_1} \cdot I(-Excep f_1))$, by PIII we have

$f_2 : \widetilde{X} \to \widetilde{X}_1$ s.t.

$f_2^{-1}(f_1^{-1} I_Y \cdot O_{\widetilde{X}_1} I(-Excep f_1)) \cdot O_{\widetilde{X}}$ is an Ideal sheaf of snc div.

$\square$