Geometry & moduli of polarized var by Birkar 21.8.21

§5 Birational boundedness on pseff pairs.

§6 Boundedness of polarized pairs

§7 Moduli of polarized pairs.

Today Thm 5.2 Fix $d \in \mathbb{Z}_{>0}$, $\delta \in \mathbb{R}_{>0}$, $\Phi \subseteq \mathbb{Q}$: DCC.

$\Rightarrow \exists\, m = m(d, \delta, \Phi)$ s.t.

if
- (X, B) : klt. dim d
- $B \in \Phi$
- N : nef big $\mathbb{R}$div
- $N - K_X - B$ & $K_X + B$: pseff
- $N = E + R$, E : integral pseff, $R \in \{0\} \cup [\delta, \infty)$

$\Rightarrow |m'N + L|$ & $|K_X + m'N + L|$: birational $\forall\, m' \geq m$, $\forall\, L$: pseff integral.

Rem: ① DO NOT assume X or (X, B) : ε-lc.
BUT $K_X + B$: pseff.

② if $N = K_X + B$: nef & big $\rightsquigarrow$ special case by HMX.

③ NOT true if (X, B) : lc.

e.g. • X : toric Fano, B : torus inv. boundary $\rightsquigarrow (X, B)$: lc, $K_X + B \sim 0$

• $N = -K_X$. $\rightsquigarrow \not\exists$ uniform m st. $|-mK_X|$: birational

---

Idea: Apply 4.2, need:
① X : ε-lc
② N : nef & big
③ $N - K_X$: pseff
④ $N = E + R$

Fix $1 \gg \varepsilon > 0$

extract all exceptional divisors with log discrepancy $< \varepsilon$

$\rightsquigarrow \begin{cases} K_{X'} + B' = \phi^*(K_X + B) \\ N' = \phi^* N \end{cases} \rightsquigarrow \begin{cases} X' : \varepsilon\text{-lc} \ \checkmark \\ N' : \text{nef \& big} \ \checkmark \\ N' - K_{X'} = N' - K_{X'} - B' + B' : \text{pseff} \ \checkmark \end{cases}$

$N' = \phi^* E + \phi^* R$. problem: How to control coefficient of exceptional divisors?

WANT $N' - {}^{\exists}\delta' \mathrm{Exc}(\phi)$: pseff.

Key: $N' = K_{X'} + B' + \text{big}$

$= \underbrace{K_{X'} + \Delta' + \text{big}}_{\text{big}} + (B' - \Delta') \geq \delta' \mathrm{Exc}(\phi)$

Prop 5.4 Fix $d \in \mathbb{Z}_{>0}$, $\Phi \subseteq \mathbb{Q}$: DCC.

$\Rightarrow \exists\, \ell = \ell(d, \Phi)$ s.t.

if
- (X, B) : lc. dim d
- $B \in \Phi \cup (\frac{\ell-1}{\ell}, 1]$
- $K_X + B$: pseff

$\Rightarrow K_X + B_{\lfloor \ell \rfloor}$: pseff.

Here $\begin{cases} b_{\lfloor \ell \rfloor} = \frac{\lfloor b\ell \rfloor}{\ell} \\ B_{\lfloor \ell \rfloor} = \frac{\lfloor \ell B \rfloor}{\ell} \end{cases}$

pf: (standard ACC + global ACC).

Assume $\nexists\, \ell \Rightarrow \forall \ell, \exists (X^\ell, B^\ell)$: lc (WMA: dlt, $\mathbb{Q}$-factorial).
- $B^\ell \in \Phi \cup (\frac{\ell-1}{\ell}, 1]$ (WMA: $B^\ell \in \mathbb{Q}$) by increasing B^ℓ
- $K_{X^\ell} + B^\ell$: pseff
- $K_{X^\ell} + B^\ell_{\lfloor \ell \rfloor}$: not pseff. $B^\ell_{\lfloor \ell \rfloor} = \Delta^\ell$

$\Psi := \{\text{coeff of } B^\ell \mid \forall \ell\}$: DCC set. $\subseteq \mathbb{Q}$.

$$\begin{array}{ccl} X^\ell & \dashrightarrow & Y^\ell \; : (K_{X^\ell} + \Delta^\ell)\text{-MFS} \\ & & \downarrow \\ & & T^\ell \end{array}$$

$(K + \Delta)$-MMP

Claim: (Y^ℓ, B_{Y^ℓ}) : lc. $\forall \ell \gg 0$

pf: if NOT lc for infinitely many ℓ,

$\Rightarrow$
- $\Delta_{Y^\ell} \leq {}^{\exists} C_{Y^\ell} \lneq B_{Y^\ell}$ with coefficient $(b^\ell - \frac{1}{\ell^2})^\ell_{\lfloor \ell \rfloor} \leq c^\ell \leq b^\ell$
- (Y^ℓ, C_{Y^ℓ}) : strictly lc

Here $\{\text{Coeff } C_{Y^\ell}\}_\ell$: DCC & $\mathrm{lct}(Y^\ell, C_{Y^\ell} - c^\ell P^\ell; P^\ell) = c^\ell < b^\ell$

$\underset{ACC}{\Rightarrow}$ such c^l is in $\begin{cases} DCC \\ ACC \end{cases}$ set $\Rightarrow$ such c^l is finite.

$\circlearrowleft$ $b^l - \frac{1}{l} < c^l < b^l + \{b^l\}$: DCC.

($\{\text{coeff } C^l\}$: DCC $\Leftarrow$ take a sequence $\{c^{l_i}\} \downarrow$. WMA $l_i \nearrow \infty$

Note $b^{l_i} - \frac{1}{l_i} < c^{l_i} \leq b^{l_i} \in \Phi$: DCC

<u>WMA</u>: $b^{l_i} \nearrow$ & $\lim b^{l_i} = b$

$\Rightarrow \lim c^{l_i} = b \geq b^{l_i} \geq c^{l_i}$ $\square$)

---

so we get $\begin{cases} (Y^l, B_{Y^l}) : \text{lc psef} \\ (Y^l, \Delta_{Y^l}) \longrightarrow Z^l : \text{MFS} \end{cases}$ $\forall l \gg 0$

$\Rightarrow$ $\Delta_{Y^l} \underset{\neq}{\overset{\exists}{\subset}} C_{Y^l} \leq B_{Y^l}$ s.t. $K_{Y^l} + C_{Y^l} \equiv 0 / Z^l$

$\Rightarrow$ $\begin{cases} (F^l, C_{F^l}) : \text{lc CY.} \\ \{\text{coeff } C_{F^l}\}_{l \gg 0} : \text{DCC} \end{cases}$ $\underset{\substack{\text{Global} \\ \text{ACC}}}{\Rightarrow}$ $\{\text{coeff } C_{F^l}\}$: finite set.

F^l: fiber gen

$\Rightarrow \{\text{coeff } C_{Y^l}\}$: finite set $\Rightarrow \{\text{coeff } B_{Y^l}\}$: finite set.

take $k \gg 0$ s.t. $kB_{Y^l} \in \mathbb{Z}$ $\forall l$

$\Rightarrow$ $kB_{Y^k} \in \mathbb{Z} \Rightarrow$ $\Delta_{Y^k} = \frac{\lfloor kB_{Y^k} \rfloor}{k} = B_{Y^k}$ $\circlearrowleft$

<u>proof of Thm 5.2</u> Take l as above.

take $X' \overset{\phi}{\to} X$ extracting all exceptional divisor E with $a(E, X, B) < \frac{1}{2l}$.

$\rightsquigarrow$ $\begin{cases} K_{X'} + B' = \phi^*(K_X + B) \\ N' = \phi^* N \\ E' = \phi^* E \\ R' = \phi^* R \end{cases}$

$\Rightarrow \forall E$: exc$/X'$.
$a(E, X') \geq a(E, X', B') \geq \frac{1}{2l}$

$\Rightarrow$ X': $\frac{1}{2l}$-lc

take $r \in \mathbb{Z}_{>0}$ s.t. <u>$r\delta \geq 1$</u>

<u>Claim</u> $(6l + 2r)N'$ satisfies 4.2

<u>More precisely</u>
- X': $\frac{1}{2l}$-lc ✓
- N': nef big ✓
- $(6l+2r)N' - K_{X'}$: pseff $\quad (N' - K_{X'} = \underbrace{N' - K_{X'} - B'} + B')$

- $(6l+2r)N' = (\text{integral \& pseff}) + (\geq 1)$

$$= \underbrace{(\lfloor 6lN' \rfloor + \lfloor 2rE' \rfloor - 2S' + J')} + (\{6lN'\} + \{2rE'\} + 2rR' + 2S' - J').$$

Here $S' = \mathrm{Supp}(\mathrm{Exc}\,\phi)$. $J' = \mathrm{Supp}(R' + S')$

<u>Assume the Claim</u> $\Rightarrow \exists n$ s.t. $|n(6l+2r)N'|$: birational.

$\Rightarrow \quad 3d\,n(6l+2r)N'$: potentially birational

HMX.

$\Rightarrow \quad 3dn(6l+2r)N$: potentially birational.

$\quad m = 3dn(6l+2r) + r + 2$

Claim: $\forall m' \geq m$, $\forall \underset{\text{integral}}{L}$: pseff, $\lfloor m'N + L \rfloor$: potentially birational.

$$\| $$

$$\underbrace{(m' - (r+2))N}_{\text{pot. bir.}} + (r+2)N + L - \{m'N\}$$

$$\| $$

$$\underbrace{2N + L + rE}_{\text{big}} + \underbrace{rR - \{m'N\}}_{\geq 0}$$

<u>To prove the Claim</u> ① $\lfloor 6lN' \rfloor + \lfloor 2rE' \rfloor - 2S' + J'$: pseff

② $\{6lN'\} + \{2rE'\} + 2rR' + 2S' - J' \geq 1$.

recall $K_{X'} + B'$: pseff $\underset{\text{prop}}{\Rightarrow}$ $\begin{cases} K_{X'} + \Delta' : \text{pseff where } \Delta' = B'_{\lfloor l \rfloor}. \\ B' - \Delta' \geq \frac{1}{2l} S' \quad \left(\begin{array}{l} \mathrm{coeff}_{S'} B' \geq 1 - \frac{1}{2l} \\ \mathrm{coeff}_{S'} \Delta' = 1 - \frac{1}{l} \end{array} \right). \end{cases}$

$$\Rightarrow N' - \frac{1}{2l} S = \underbrace{N' - K_{X'} - B'}_{\text{big}} + \underbrace{(B' - \Delta' - \frac{1}{2l} S)}_{\geq 0} + \underbrace{K_{X'} + \Delta'}_{\text{pseff}} : \text{big}.$$

$$① = \underbrace{6l(N' - 3S') + 2rE'}_{\text{big}} + \underbrace{S' - \{2rE'\}}_{\geq 0} + \underbrace{J' - \{6lN'\}}_{\geq 0}.$$

$(\mathrm{Supp}\{2rE'\} \subseteq \mathrm{Exc}(\phi), \ \mathrm{Supp}\{6lN'\} \subseteq \mathrm{Supp}(\mathrm{exc} + R').)$

$② = \{6N'\} + \{2rE'\} + 2rR' + 2S' - J' \geq \underline{2rR' + 2S' - J'} \geq J'$

& $\mathrm{Supp}\,② = J'$

Thm 6.1. Fix $d \in \mathbb{Z}_{>0}$. $v, \varepsilon, \delta \in \mathbb{R}_{>0}$. 21.8.20

$\Rightarrow \{(X, \mathrm{Supp}\,B)\}$ log bounded where

- (X,B) ε-lc dim d.
- $B \in \{0\} \cup [\delta, +\infty)$
- $K_X + B$: nef
- N : nef big $\mathbb{R}$-div
- $N = E + R$, E : int. Psef, $R \in \{0\} \cup [\delta, +\infty)$.
- $(K_X+B+N)^d \leq v$.

if $N \geq 0 \Rightarrow \{(X, \mathrm{Supp}(B+N))\}$: log bounded.

Idea: by 4.3, $\exists m$, s.t. $|\underset{\underset{M.}{\|}}{m(K_X+B+N)}|$: birational, $M^d \leq \square$

$\rightsquigarrow (X, B+M)$: log birationally bounded

Key: $\exists t > 0$ uniform s.t. $(X, B+tM)$: $\frac{\varepsilon}{2}$-lc.

$\xrightarrow[\text{HMX}]{}$ $(X, B+M)$: log bounded. (if K_X+B+M : ample !!)

Pf of 6.1 Step 1 Construct M s.t. $\begin{cases} |M| : \text{birational} \\ M \geq 1 \rightsquigarrow \text{bound frac part.} \\ M^d \leq \square \quad \mathrm{Supp}(M) \supseteq \mathrm{Supp}\,B. \end{cases}$

by 4.3

$\exists m, l$ s.t. (WMA $m\delta \geq 1$) $\ni L$

$|mK_X + mlN \quad + mE|$: birational.

$M := L + mB + mR \sim m(K_X+B) + (ml+m)N$

$\Rightarrow$ ① $|M|$: birational as $M \geq L$

② $M \geq 1$ as $M \geq mB + mR$ & $\mathrm{Supp}\{M\} \subseteq \mathrm{Supp}(B+R)$

③ $M^d \leq \square$.

Recall: M: nef & big $\mathbb{R}$-div.

From now we assume M: ample.

Step 2 Show $(X, \underset{\substack{\| \\ \operatorname{Supp} M}}{\underline{\operatorname{Supp}(B+M)}})$: log bounded.

Recall: ([9, 4.4])

if
- X: normal proj dim d
- $B \in \{0\} \cup [\delta, +\infty)$
- $A \geq 0$ nef $\mathbb{Q}$-div. $|A|$: birational
- $A - (K_X+B)$: pseff.
- $A^d < v$
- $B + A \geq \operatorname{Supp}(A)$

$\Rightarrow \exists \mathcal{P}$: bounded family of log sm pairs $(\overline{X}, \overline{\Sigma})$ (dep on d, δ, v).

$$\begin{array}{ccc} & W & \\ \phi \swarrow & & \searrow \psi \\ X & & \overline{X} \end{array}$$

s.t.
- $X \dashrightarrow \overline{X}$ bir
- $\overline{\Sigma} = \operatorname{Supp}\left((B+A)_{\overline{X}} + \operatorname{Exc}(\overline{X}/X) \right)$.
- $\overline{A} := \psi_* \phi^* A \leq {}^{\exists} C$

Now we may find A: ample $\mathbb{Q}$-div s.t.
$$\begin{cases} \frac{A}{2} \leq M \leq A. \\ \left| M - \frac{A}{2} \right| \ll 1. \\ \operatorname{Supp}(A) = \operatorname{Supp}(M). \end{cases}$$
to apply [9, 4.4]

Step 3 Show $\operatorname{lct}(X, B; M) \geq {}^{\exists} t > 0$ uniform.

Assume $(X, B+tM)$: not klt

$\rightsquigarrow$ $(X, B+tA)$ not klt (K_X+B+tA : ample)

$$\psi_* \phi^* (K_X+B+tA) = K_{\overline{X}} + \overline{B} + t\overline{A}$$

$\xrightarrow{\text{negativity}}$ $\psi^*(K_{\overline{X}} + \overline{B} + t\overline{A}) \geq \phi^*(K_X+B+tA)$

$\Rightarrow$ $(\overline{X}, \underbrace{\overline{B}}_{\leq (1-\varepsilon)\overline{\Sigma}} + t\overline{A})$ not klt. $\Rightarrow$ $\underbrace{(\overline{X}, (1-\varepsilon)\overline{\Sigma}}_{\text{log bounded}} + t\overline{A})$ not klt

$$t \geq \frac{\varepsilon}{2e} > 0$$
$\Uparrow$

<u>Step 4</u> Show $(X, \mathrm{Supp}(B+M))$ log bounded.

note that
$$\begin{cases} \circ\ (X,B): \varepsilon\text{-lc} \\ \circ\ (X, B+\frac{t}{2}M): \frac{\varepsilon}{2}\text{-lc} \\ \circ\ K_X+B+\frac{t}{2}M : \text{ample} \\ \circ\ B+\frac{t}{2}M \geq \min\{\delta, \frac{t}{2}\} > 0 \\ \circ\ (X, \mathrm{Supp}(B+M)) \text{ log bir bounded (Step 3)} \end{cases} \xrightarrow[\text{HMX}]{} \text{log bounded.}$$

<u>Step 5</u> If M is only nef & big ($M - K_X - B$: nef & big)

$\rightsquigarrow$ BPF thm $X \xrightarrow{\exists \pi} Y$ birational

$M = \pi^* M_Y$ $\quad M_Y$: ample.

$K_X + B = \pi^*(K_Y + B_Y)$

$N = \pi^*(N_Y)$.

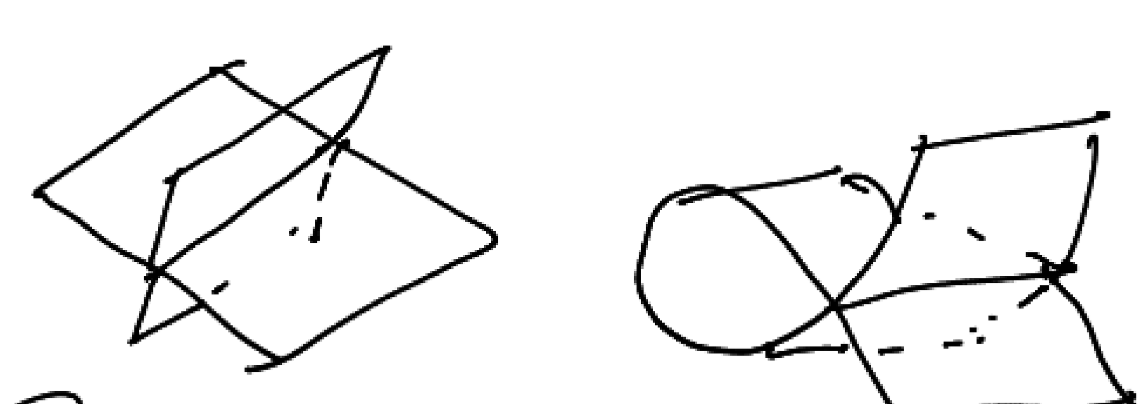
$\underset{\text{Step 4}}{\rightsquigarrow}$ $(Y, B_Y + \frac{t}{2}M_Y)$ $\frac{\varepsilon}{2}$-lc & log bounded

$\underset{\text{CY fibration}}{\rightsquigarrow}$ $(X, B+M)$ log bounded. (if $N \geq 0$, WMA $M \geq N$)

<u>Cor 1.6</u> Fix $d \in \mathbb{Z}_{>0}$, Φ : DCC, $v \in \mathbb{R}_{>0}$

$$\Rightarrow \left\{ (X, \mathrm{Supp}\, B) \;\middle|\; \begin{array}{l} \circ\ (X,B) : \text{klt CY} \\ \circ\ B \in \Phi \ (\Rightarrow (X,B) : \exists\varepsilon\text{-lc}) \\ \circ\ N : \text{nef big integral} \\ \circ\ N^d \leq v \end{array} \right\} : \text{log bounded}$$

<u>Next:</u> slc Calabi-Yau



<u>Definition</u>

$\circ$ X : S_2 with nodal in codim 1

$\circ$ B : div on X. s.t. no comp of $B \subseteq \mathrm{Sing}(X)$

$\circ$ $K_X + B$: $\mathbb{R}$-Cartier.

$\circ$ $\nu : X^\nu \to X$. $\quad K_{X^\nu} + B^\nu = \nu^*(K_X + B)$

(X^ν, B^ν) : lc $\underset{\text{def}}{\Longleftrightarrow}$ (X,B) : slc polarized slc CY : (X, B), N

$\overset{\text{def}}{\iff}$
- $K_X + B \sim_{\mathbb{R}} 0$
- $(X, B+tN)$: slc, $\exists t>0$
- N : integral ample ≥ 0.

Cor 1.8 Fix $d \in \mathbb{Z}_{>0}$, $v \in \mathbb{R}_{>0}$, Φ : DCC.

$$\Rightarrow \left\{ (X, B, N) \;\middle|\; \begin{array}{l} \circ (X,B), N : \text{pol. slc CY dim } d \\ \circ B \in \Phi \\ \circ \mathrm{vol}(N) = v \end{array} \right\} : \text{bounded family.}$$

pf: if $\exists t>0$ uniform s.t. $(X, B+tN)$: slc.

then
- coeff$(B+tN) \in$ DCC.
- $\mathrm{vol}(K_X+B+tN) = t^d \cdot v$ fixed HMX

$\Longrightarrow$ log bounded.

Thm 6.4 Fix $d \in \mathbb{Z}_{>0}$, $v, \delta \in \mathbb{R}_{>0}$, Φ : DCC.

$\Rightarrow \exists t = t(d, v, \delta, \Phi) > 0$ s.t.

if
1. (X, B) : slc CY. dim d
2. $B \in \Phi$
3. $N \geq \delta$ nef $\mathbb{R}$-div.
4. $(X, B+uN)$: slc for some $u>0$
5. $N|_S$ big & $\mathrm{vol}(N|_S) \leq v$ $\forall S \subseteq X$ irr comp.

then $(X, B+tN)$: slc.

Idea: slc $\underset{\text{normalization}}{\rightsquigarrow}$ lc $\underset{\text{dlt}}{\rightsquigarrow}$ dlt $\underset{\text{ext}}{\rightsquigarrow}$ ϵ-lc (use 4.2 $\rightsquigarrow$ birationally bdd) & step 3 in 6.2)

pf: take normalization $\left\{ \begin{array}{l} K_{X^\nu} + B^\nu = \nu^*(K_X+B) \\ N^\nu = \nu^* N \end{array} \right.$ $B^\nu \in \Phi \cup \{1\}$.

WMA: $\left\{ \begin{array}{l} X : \text{normal \& irreducible} \\ (X, B+uN) : \text{lc.} \end{array} \right.$

take dlt modification $\phi : X' \to X$.

$$④ \Rightarrow \phi^* N = \phi_*^{-1} N \Rightarrow ②$$

<u>WMA</u>: $\begin{cases} (X, B) : \text{dlt.} \\ X : \mathbb{Q}\text{-factorial klt} \end{cases}$

<u>Recall</u> Global ACC $\Rightarrow \exists \varepsilon > 0$ (uniform) s.t.

if $a(E, X, B) < \varepsilon \overset{(\star)}{\Rightarrow} a(E, X, B) = 0$

extracting all E by $\phi : X' \to X$. such that $a(E, X, 0) < \varepsilon$

$(\star) + ④ \Rightarrow \phi^* N = \phi_*^{-1} N$ $\left(\begin{array}{l} \Rightarrow a(E, X, B) < \varepsilon \\ \Rightarrow a(E, X, B) = 0 \end{array} \right)$

$(\star) \Rightarrow B' \in \Phi \cup \{1\}$

$\rightsquigarrow$ <u>WMA</u>: $X : \varepsilon$-lc

<u>Goal</u>: $\operatorname{lct}(X, B, N) \geq t$

Suppose $K_X + B + tN$: not klt. (nef)

$\xrightarrow{\text{neg. lem}} K_{\overline{X}} + \overline{B} + tN_{\overline{X}}$: not klt.

recall $\forall P \in \operatorname{Supp} \overline{B}$

either $\begin{cases} \operatorname{mult}_P \overline{B} \leq 1 - \varepsilon \\ \text{or } \operatorname{mult}_P \overline{B} = 1 \ \& \ \operatorname{mult}_P \overline{N} = 0 \end{cases}$

by definitn of ε

$\Rightarrow t \geq \dfrac{\varepsilon}{c}.$

<u>Claim</u> $(X, \operatorname{Supp}(B+N))$ birationally log bounded.

i.e $\exists (\overline{X}, \Sigma)$ log bounded family

- $\Sigma = \operatorname{Supp}(B_{\overline{X}} + N_{\overline{X}} + \operatorname{Exc}(\overline{X}/X))$
- $N_{\overline{X}} \leq \exists c$.

<u>to prove Claim</u> we need to find

M s.t. $\begin{cases} \circ \ |M| : \text{birational, nef. } \mathbb{Q}\text{-div.} \\ \circ \ M \geq N \\ \circ \ B + M \geq \operatorname{Supp} M \\ \circ \ M^d \leq \square \end{cases}$

by 4.5, $\exists\, m, l$ s.t. $|mK_X + lN|$: birational
$$\underset{L}{\rotatebox{90}{$\in$}}$$

$\rightsquigarrow M := L + mN + mB \qquad (m\delta \geq 1)$

Moduli of slc pol CY $\hfill$ 21. 8. 24.

Moduli functor & coarse moduli space

$\mathcal{P}$: a set of geometric objects

$F : (\mathrm{Red})\mathrm{Sch} \longrightarrow \mathrm{Sets}$

$\qquad S \longrightarrow \{ f : X \to S \text{ flat},\ X_s \in \mathcal{P} \}$ with certain properties.

if $T \to S$

$\rightsquigarrow F(S) \longrightarrow F(T)$

$$\begin{tikzcd} X \arrow[d, "f"'] & X \times_S T \arrow[l] \arrow[d, "f_T"] \\ S \arrow[ur, phantom, "\square"] & T \arrow[l] \end{tikzcd}$$

If F is representable by $M \in \mathrm{Sch}$. $\quad$ fine moduli

then $F(S) \cong \mathrm{Hom}(S, M)$. $\rightsquigarrow F(M) \cong \mathrm{Hom}(M, M)$
$$\underset{(U \to M)}{\rotatebox{90}{$\in$}} \qquad \underset{(\mathrm{id})}{\rotatebox{90}{$\in$}}$$

$$\rightsquigarrow \begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ S \arrow[r, "\alpha"] & M \end{tikzcd}$$

$$\begin{array}{ccccc} (X \to S) \in F(S) & \cong & \mathrm{Hom}(S, M) & \ni & \alpha \\ \uparrow & & \uparrow & & \uparrow \\ (U \to M) \in F(M) & \cong & \mathrm{Hom}(M, M) & \ni & \mathrm{id} \end{array}$$

F is coarsely representable

if $\exists\, M \in \mathrm{Sch}$ & $\eta : F \longrightarrow \mathrm{Hom}(-, M)$ natural transform.

s.t. (1) $\eta_{\mathrm{Spec}\,k} : F(\mathrm{Spec}\,k) \xrightarrow{\cong} \mathrm{Hom}_k(\mathrm{Spec}\,k, M) = M(k)$

(2) $\forall\, S \in \mathrm{Sch}$ & $\xi : F \to \mathrm{Hom}(-, S)$

$\exists!\ \mathrm{Hom}(-, M) \xrightarrow{\nu} \mathrm{Hom}(-, S)$

s.t. $\nu \circ \eta = \xi$.

Coarse moduli ⇝ (1) $\Leftrightarrow$ $\{\text{closed point of } M\} \overset{(1)}{=} P$.

(2) $\forall S, \exists \eta: F(S) \longrightarrow \mathrm{Hom}(S, M)$.

⇝ $(U \to S) \longmapsto f_U : S \to M$.

Stable pair (X, Δ) : (1) proj. geom. conn. slc$/K$

(2) $K_X + \Delta$: ample

Stable family $(X, \Delta) \xrightarrow{f} S$: (1) f : flat proj. with reduced geom. conn S_2 nodal in codim 1 fibers.

(2) $\Delta = \sum a_i D_i$ D_i : Mumford div.

(3) $K_{X/S} + \Delta$: $\mathbb{Q}$-Cartier

(4). $\forall s \in S$, (X_s, Δ_s) : slc, $K_{X_s} + \Delta_s$: ample

Mumford divisor : D is Mumford$/S$ if $\exists U \subseteq X$ s.t.

- $\operatorname{codim}(X_s \setminus U_s) \geq 2$ $\forall s \in S$
- $D|_U$: relative Cartier (Cartier + $D \not\supseteq U_s$.)
- $D = \overline{D|_U}$
- $X \to S$ is smooth at gen points of $X_s \cap D$ $\forall s \in S$.

⇝ good to say $D|_{X_s}$.

Def S : reduced sch.

(d, c, v)-pol. CY family$/S$ $X \to S$.

- $(X, B + uN) \to S$ stable family $\exists u > 0$. $d = \dim(X/S)$
- $B = cD$. $D \geq 0$ Mumford$/S$
- $N \geq 0$ Mumford$/S$
- $K_{X/S} + B \sim_{\mathbb{Q}} 0 /S$
- $\operatorname{vol}(N|_{X_s}) = v$ $\forall s \in S$.

$\mathcal{PCY}_{d,c,v}$: Red-Sch $\longrightarrow$ Sets.

$S \longmapsto \mathcal{PCY}_{d,c,v}(S) = \{X \to S \ (d,c,v)\text{-pol. CY}\}/\cong$

<u>Thm</u> $\mathcal{PCY}_{d,c,v}$ has a <u>projective</u> coarse moduli ($PCY_{d,c,v}$)

<u>Idea:</u> $(X,B),N$: (d,c,v)-pol CY. $\Rightarrow$ log bounded

$\rightsquigarrow X \hookrightarrow \mathbb{P}^n$.

we need to show $\exists$ fine moduli for $(X \subseteq \mathbb{P}^n, B, N)$

$(E^S PCY_{\Xi})$

$PCY_{d,c,v,n} := E^S PCY_{\Xi} / PGL_{n+1}$: coarse moduli, <u>separated, proper proj</u>

Strongly embedded (d,α,v,n)-marked stable family.

$$f : (X \subseteq \mathbb{P}^n_S, \Delta) \longrightarrow S$$

- $f : X \to S$ stable family of rep dim d.
- $\Delta = \sum a_i D_i \quad \alpha = (a_1, \cdots, a_m)$, $(K_{X_s} + \Delta_s)^d = v$. $\forall s$.
- $$\begin{tikzcd} X \arrow[r, hook, "g"] \arrow[dr, "f"'] & \mathbb{P}^n_S, \mathcal{O}(1) \arrow[d, "\pi"] \\ & S \end{tikzcd} \quad \mathcal{L} = g^*\mathcal{O}(1)$$
- $R^q f_* \mathcal{L} \cong R^q \pi_* \mathcal{O}(1) = \begin{cases} \pi_* \mathcal{O}(1) & q = 0 \\ 0 & q > 0 \end{cases}$

$\mathcal{E}^S\mathcal{MLSP}_{d,\alpha,v,n}(S) = \{ f : (X \subseteq \mathbb{P}^n_S, \Delta) \to S \}$.

<u>Thm</u> (Kollár) $\mathcal{E}^S\mathcal{MLSP}_{d,\alpha,v,n}$ has a fine moduli $E^S MLSP_{d,\alpha,v,n}$.

<u>Step 1</u> Find (α', v', n) s.t

$\{(d,c,v)\text{-pol CY families}\} \subset \{(d,\alpha,v',n)\text{-}E^S MLSP \text{ families}\}$

$f : (X,B),N \to S$

$\rightsquigarrow \exists t = t(d,c,v)$ s.t
$\exists r = r(d,c,v)$

- $K_{X/S} + B + tN$: ample$/S$.
- $(X_s, B_s + tN_s)$: slc $\Rightarrow$ $N \leq k$?
- $r(K_{X/S} + B + tN)$ relative very ample$/S$.
- $R^j f_* \mathcal{O}_X(m r(K_{X/S} + B + tN)) = 0 \ \forall m > 0, j > 0$

$\rightsquigarrow$

- $X \overset{g}{\hookrightarrow} (\mathbb{P}^n_S, \mathcal{O}(1))$.
- $n = h^0(r(K_{X_s} + B_s + tN_s))$.
- $\mathcal{L} = g^*\mathcal{O}(1) = r(K_{X/S} + B + tN)$

α = coeffs of $(B + tN)$

$v' = t^d v$

(1:1) $\leftarrow$ if $\Delta = \sum \alpha_i D_i$ $\quad \alpha_i \in c\mathbb{Z} + t\cdot\{0,1,\dots,k\}$. ok if take $t < \frac{c}{k}$

then Δ has a unique way to write

as $\Delta = B + tN$ where $B \in c\mathbb{Z}$, $N \in \{0,1,\dots,k\}$

Now denote $\Xi = (d, c, v, t, r, n)$

strongly emb. Ξ-polarized slc CY family$/S$

is $f : (X \subseteq \mathbb{P}^n_S, B), N \to S$ as above.

$\rightsquigarrow$ moduli functor $\mathcal{E}^s PCY_\Xi$

Prop 7.8 $\mathcal{E}^s PCY_\Xi$ has a fine moduli

Idea:

$$\begin{tikzcd}
(X, B+tN) \arrow[r] \arrow[d] \arrow[rr, bend left] & U' \arrow[r] \arrow[d] & (X^{(0)} \subseteq \mathbb{P}^n_M, \Delta^{(0)} = cD^{(0)} + tN^{(0)}) \arrow[d] \\
S \arrow[r] \arrow[rr, bend right] & M' \arrow[r] & M^{(0)}
\end{tikzcd}$$

$\mathcal{E}^s PCY_\Xi(S) \ni$

$\wr\wr$

$\mathcal{E}^s MSLP_{(d,\alpha,v,n)} \rightsquigarrow$ fine moduli $M^{(0)}$

Goal $\exists (U' \to M') \in \mathcal{E}^s PCY_\Xi$ factor through $S \to M$

recall $\{$fibers of $X^{(0)} \to M^{(0)}\} \overset{1:1}{\longleftrightarrow} \{$
- (X_s, Δ_s) : slc. dim d
- $\Delta_s = \sum \alpha_i D_i \overset{\exists!}{=} B_s + tN_s$
- $K_{X_s} + \Delta_s$: ample
- $(K_{X_s} + \Delta_s)^d = v' = t^d v$

BUT: (X_s, B_s) may not CY

- N_s may not ample.
- N_s not nec. $\mathbb{Q}$-Cartier!

∘ [Kollár] $\exists\, M^{(3)} \to M^{(0)}$ locally closed partial decomposition

s.t. ∘ $(X^{(3)} \subseteq \mathbb{P}_{M^{(3)}}, \Delta^{(3)} = cD^{(3)} + tN^{(3)}) \to M^{(3)}$

∘ $(X^{(3)}, cD^{(3)}) \to M^{(3)}$ $\mathbb{Q}$-Cart on fibers

∘ $S \to M^{(3)} \to M^{(0)}$

Consider $M^{(4)} \subseteq M^{(3)}$
$$\begin{array}{c} M^{(4)} \\ \parallel \\ \{s \mid K_{X_s}+cD_s \sim_{\mathbb{Q}} 0\}. \end{array}$$

<u>Claim:</u> $M^{(4)}$: locally closed.

$$K_{X^{(4)}} + cD^{(4)} \sim_{\mathbb{Q}} 0 \,/ M^{(4)}.$$

$\rightsquigarrow \begin{cases} \circ\ (X^{(4)}, cD^{(4)}+tN^{(4)}) \longrightarrow M^{(4)} \\ \circ\ S \longrightarrow M^{(4)} \longrightarrow M^{(3)}. \end{cases}$

<u>Finally</u> recall $\exists r.$ $r(K_{X_s}+\Delta_s)$: Cartier $\forall s \in M^{(4)}$

$\overset{\rightsquigarrow}{\text{Kollár}}$ $r(K_{X^{(4)}/M^{(4)}}+\Delta^{(4)})$: Cartier

we consider $M^{(5)}$ be the set $s \in M^{(5)}$ s.t
$$\mathcal{O}_{X_s}(1) \cong \mathcal{O}_{X_s}(rK_{X_s}+\Delta_s)).$$

$\rightsquigarrow \begin{cases} M^{(5)} \text{ locally closed.} \\ S \rightarrow M^{(5)} \rightarrow M^{(4)} \end{cases}$

$$M^{(5)} = \mathcal{EPCY}_{\Xi}.$$

<u>Lem 7.5</u> $(X,B) \to S$ locally stable $\Leftrightarrow \begin{cases} \circ\ f\text{: flat}, \ldots \\ \circ\ (X_s, B_s)\text{: slc.} \end{cases}$

$S' := \{s \in S \mid (X_s, B_s) : \text{slc CY}\}$

$\Rightarrow \begin{cases} S' : \text{locally closed subset.} \\ K_{X'/S'}+B' \sim_{\mathbb{Q}} 0 / S' \end{cases}$

<u>Pf:</u> We may assume $\overline{S'} = S$. (we want $K_{X/S}+B \sim_{\mathbb{Q}} 0/S$)

by noetherian induction, we may replace S by open affine subset.

<u>Lem 7.4</u> if $\circ$ X: normal, $B \geq 0$ $\mathbb{Q}$-div
- $\circ$ $X \to S$
- $\circ$ $T \subseteq S$ dense s.t. (X_s, B_s) : lc CY $\forall s \in T$

$\Rightarrow \exists U \subseteq S$ open s.t. $K_{X/S}+B \sim_{\mathbb{Q}} 0/U$ & (X,B) : lc $/U$.

7.4⇒7.5) ∘ WMA: S: smooth affine.

∘ (X, B) : slc $\Leftarrow$ (X_s, B_s) : slc

$\Downarrow$

∘ (X^ν, B^ν) : lc., $K_{X^\nu_s} + B^\nu_s \sim_{\mathbb{Q}} 0$

$\parallel$

$\coprod (X_i, B_i)$

$\Rightarrow \exists\, U \underset{\text{s.t.}}{\subseteq} S$ open. $K_{X_i} + B_i \sim_{\mathbb{Q}} 0 / U$.

$\overset{7.4}{\underset{\text{gluing}}{\Longrightarrow}}$ $K_{X^\nu} + B^\nu \sim_{\mathbb{Q}} 0 / U$.

Pf of 7.4 ∘ WMA: S: smooth

∘ $(X', B') \xrightarrow{\phi} (X, B) \to S$

$\parallel$

$\phi^{-1}_* B + \mathrm{Exc}$.

∘ WMA: (X'_s, B'_s) : log smooth. by shrinking S.

∘ $K_{X'_s} + B'_s = \phi^*(K_{X_s} + B_s) + E_s \sim_{\mathbb{Q}} \underset{\text{exc}/X_s}{E_s \geq 0}\ \forall s \in T$.

take $\begin{cases} A' : \text{ample}/S \text{ on } X' \\ r(K_{X'} + B') \in \mathbb{Z}. \end{cases}$

$\Rightarrow \forall m, l \in \mathbb{Z}_{>0}$ by USC

$h^0(mr(K_{X'_s} + B'_s) + lA'|_{X'_s}) \geq h^0(mr(K_{X'_\eta} + B'_\eta) + lA'|_{X'_\eta})$.

$\Rightarrow \kappa_\sigma(K_{X'_\eta} + B'_\eta) \leq \kappa_\sigma(K_{X'_s} + B'_s) \underset{\text{e}}{=} 0 \quad s \in T$.

if $\kappa_\sigma(K_{X'_\eta} + B'_\eta) = -\infty \Rightarrow K_{X'} + B'$ not psef. $/S$.

$\overset{\text{MMP}}{\Longrightarrow}$ $K_{X'_s} + B'_s$ not psef $\forall s \in S$ general ↯

if $\kappa_\sigma(K_{X'_\eta} + B'_\eta) = 0 \underset{\text{Abundance}}{\Longrightarrow} h^0(mr(K_{X'_\eta} + B'_\eta)) \neq 0$

[Gongyo] $\Rightarrow K_{X'} + B' \sim_{\mathbb{Q}} D' \geq 0 / S$.

$\Rightarrow D'_s = E_s \quad \forall s \in T$

$\Rightarrow D' : \mathrm{exc}/X.$

$\Rightarrow K_X + B \sim_{\mathbb{Q}} \phi_* D = 0/S.$ $\square$.