Chapter 5. Special BAB

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1. Introduction

The goal of this chapter is to prove the following theorem, which is known as the Special BAB. All contents in this chapter are based on [**1**, Section 5].

Theorem 1.1. (Special BAB, [**1**, Theorem 1.4]) Let $d \in \mathbb{N}$ and $\epsilon, \delta \in \mathbb{R}_{>0}$. Consider projective varieties X satisfying:

- (X, B) is ϵ-lc of dimension d for some boundary B,
- B is big and $K_X + B \sim_{\mathbb{R}} 0$, and
- the coefficients of B are at least δ.

Then the set of such X forms a bounded family.

Hacon and Xu, [**3**, Theorem 1.3], proved Theorem 1.1 assuming the coefficients of B belong to a fixed DCC set of rational numbers, relying on the special case when $-K_X$ is ample [**2**, Corollary 1.7]. We will need their result in the proof of the theorem. To be specific, we will show that K_X has a klt n-complement. The theorem can be viewed as a special case of the Borisov–Alexeev–Borisov conjecture. In fact, if we take $\delta = 0$ in Theorem 1.1, it is equivalent to the Borisov–Alexeev–Borisov conjecture, which is the main theme of this book.

Now, we briefly explain the idea of the proof of Theorem 1.1.

- Run a $(-K_X)$-MMP and take the canonical model, by applying Theorem 3.1 ([**3**, Theorem 1.3]), we reduce to the case that X is Fano.
- By Theorem 3.2 ([**2**, Theorem 1.6]), we only need to show the log birational boundedness of (X, B).
- By Proposition 4.2 of Chapter 4, $|-mK_X|$ defines a birational map for some m depending only on d. In order to show the log birational boundedness, we can apply Lemma 3.1 of Chapter 4. The only remain thing is to show that $\mathrm{vol}(-K_X)$ is bounded from above.
- Finally, in order to show that $\mathrm{vol}(-K_X)$ is bounded from above, the idea is to construct isolated non-klt centers by $-K_X$. In fact, if $\mathrm{vol}(-K_X)$ is large enough, then we can construct an effective $\mathbb{Q}$-divisor $\Delta \sim_{\mathbb{Q}} -aK_X$ where $a \in \mathbb{Q}_{>0}$ is sufficiently small, so that (X, Δ) has a non-klt center G containing x. By Shokurov-Kollár connectedness principle [**4**, 17.4],

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the dimension of G is positive. Then we can use the method similar to the proof of Proposition 4.1 of Chapter 4 to conclude an upper bound of $\operatorname{vol}(-K_X)$.

The last part is the most technical result in this chapter.

Proposition 1.2. Let $d \in \mathbb{N}$, and $\epsilon, \delta \in \mathbb{R}_{>0}$. Then there is a number v depending only on d, ϵ, δ such that for any X as in Theorem 1.1, $\operatorname{vol}(-K_X) < v$.

2. Boundedness of volumes

In this section we will prove Proposition 1.2.

Proof of Proposition 1.2. Take $a > 0$ to be the real number such that $\operatorname{vol}(-aK_X) = (4d)^d + 1$. It suffices to prove that a is bounded from below.

Step 1. Reduce to the case when X is Fano of dimension $d \geq 2$ and B is a $\mathbb{Q}$-boundary.

When $d = 1$, the proposition clearly holds. We may assume that $d \geq 2$. Let $X' \to X$ be a $\mathbb{Q}$-factorialization. Note that X' is of Fano type. Running a $(-K_{X'})$-MMP, we get a $(-K)$-minimal model $X_{\min}$. Since B is big, $-K_{X_{\min}}$ is nef and big. By the Basepoint-free Theorem (cf. [**7**, Theorem 3.3]), $-K_{X_{\min}}$ is semiample, and defines a birational contraction $X_{\min} \to X_{\text{can}}$, where $-K_{X_{\text{can}}}$ is ample. By Exercise 4.1, $(X_{\text{can}}, B_{\text{can}})$ is ϵ-lc. By Exercise 4.2, $\operatorname{vol}(-K_X) = \operatorname{vol}(-K_{X_{\text{can}}}) = (-K_X)^d$, hence by replacing (X, B) by $(X_{\text{can}}, B_{\text{can}})$, we may assume that X is Fano. In particular, $a \in \mathbb{Q}_{>0}$. Moreover, by modifying B, we may assume that B is a $\mathbb{Q}$-boundary (cf. Exercise 4.3).

Step 2. Construct a family of non-klt centers G, such that $\operatorname{vol}(-K_X|_G)$ is bounded from above.

Fixed a smooth point $y' \in X$, since $\operatorname{vol}(-\frac{a}{2}K_X) > d^d$, by [**6**, 6.1], we may find $\Delta_1 \sim_{\mathbb{Q}} -\frac{a}{2}K_X$, such that $\operatorname{mult}_{y'}\Delta_1 > d$, and (X, Δ_1) is not lc at y'. Since $\operatorname{vol}(-\frac{a}{2}K_X) > (2d)^d$, by [**1**, 2.31(2)] (see Section 1 of Chapter 9), there is a bounded family of subvarieties of X such that for a general point x in X, there is a member G of the family and an effective $\mathbb{Q}$-divisor $\Delta_2 \sim_{\mathbb{Q}} -\frac{a}{2}K_X$ such that (X, Δ_2) is lc near x with a unique non-klt place whose center contains x and that center is G. Here we always consider general x such that $x \notin \operatorname{Supp}(\Delta_1)$, hence $(X, \Delta := \Delta_1 + \Delta_2)$ is lc near x with a unique non-klt place whose center contains x and that center is G. Recall that this family is given by finitely many morphisms $V^j \to T^j$ of projective varieties with surjective morphisms $V^j \to X$ and G is a general fiber of one of $V^j \to T^j$, and we can assume that for each j the points on T^j corresponding to the G are dense.

Denote $k := \max\{\dim V^j - \dim T^j\}$. We show that a is bounded from below by the induction on k. When $k = 0$, then $G = \{x\}$ is an isolated non-klt center and x, y' belong to two disconnected non-klt centers. By Shokurov-Kollár connectedness principle [**4**, 17.4], $-(K_X + \Delta) \sim -(1-a)K_X$ cannot be ample. Hence $a \geq 1$. Now we may assume that $k > 0$, let $b \in \mathbb{N}$ to be the smallest number such that $\operatorname{vol}(-bK_X|_G) \geq d^d + 1$ for all general G with $\dim G > 0$. We can assume that the equality is obtained for G which are general fibers of the morphism $V^j \to T^j$ for some j. By [**1**, 2.31(2)] (see Section 1 of Chapter 9), after replacing a by $a + b$ and replacing Δ, we can replace each family of G with a family of centers of strictly smaller dimensions. By the induction, we may assume that $a + b > 2\mu$ for some $\mu > 0$ depending only on d, ϵ, δ. We may assume that $a < \mu$, otherwise we are done. Therefore $b \geq \mu$. From now on, we consider general G as a general fiber of the morphism $V^j \to T^j$ and by the construction, $\mathrm{vol}(-K_X|_G) = \frac{1}{b^k}(d^d+1) \leq \frac{1}{\mu^d}(d^d+1)$ which is bounded from above.

Step 3. Construct a bounded family.

Take F to be the normalization of G. By [**1**, Theorem 3.10] (see Section 2 of Chapter 9) and the ACC for log canonical thresholds [**2**, Theorem 1.1], there is an effective $\mathbb{Q}$-divisor Θ_F with coefficients in a DCC set Φ depending only on d such that we may write

$$(K_X + \Delta)|_F \sim_{\mathbb{R}} K_F + \Delta_F = K_F + \Theta_F + P_F,$$

where P_F is pseudo-effective. By increasing a and adding to Δ, we may assume that P_F is effective and big. Since G is general, by [**1**, Lemma 3.12] (see Section 2 of Chapter 9), we may write $K_X|_F = K_F + \Lambda_F$ for some sub-boundary Λ_F such that (F, Λ_F) is sub-ϵ-lc and $\Lambda_F \leq \Theta_F \leq \Delta_F$.

By [**1**, Proposition 4.9] (see Proposition 4.2 of Chapter 4), there exists a natural number m depending only on d, ϵ, δ, such that $|-mK_X|$ defines a birational map. Take $\phi : W \to X$, A_W, R_W, Δ_m as in Notation 3.2 of Chapter 4. Take a log resolution $f : F' \to F$ of (F, Δ_F) such that the induced map $F' \dashrightarrow W$ (which is well-defined since G is general) is a morphism. Denote $A_{F'} := A_W|_{F'}$ which is base point free and defines a birational map on F'. Denote $M := \frac{1}{\delta}B$, $M_F := M|_F$. Note that $m\delta f^*M_F \sim_{\mathbb{R}} (A_W + R_W)|_{F'} \geq A_{F'}$.

Take $\Sigma_{F'}$ to be the sum of the strict transform of $\mathrm{Supp}(M_F + \Theta_F)$, and f-exceptional divisors. Fix a rational number $\epsilon' \in (0, \epsilon)$ such that $\epsilon' < \min \Phi_{>0}$. By the definition of Φ, $\mathrm{Supp}(\Theta_F) \leq \frac{\Theta_F}{\epsilon'}$. Note that by [**1**, Lemma 3.11], $\mathrm{Supp}(M_F) \leq \Theta_F + M_F$ since the coefficients of M are at least 1. Recall that by [**1**, Lemma 2.46], $K_{F'} + (2k+1)A_{F'}$ is big (see Section 3 of Chapter 2). Hence

$$\begin{aligned}
&\mathrm{vol}(K_{F'} + \Sigma_{F'} + 2(2k+1)A_{F'}) \\
\leq &\mathrm{vol}(K_{F'} + \Sigma_{F'} + 2(2k+1)A_{F'} + {\epsilon'}^{-1}(K_{F'} + (2k+1)A_{F'})) \\
\leq &\mathrm{vol}(K_F + \Sigma_F + 2(2k+1)A_F + {\epsilon'}^{-1}(K_F + (2k+1)A_F)) \\
\leq &\mathrm{vol}((1 + {\epsilon'}^{-1})K_F + \mathrm{Supp}(M_F) + \mathrm{Supp}(\Theta_F) + (2 + {\epsilon'}^{-1})(2k+1)A_F) \\
\leq &\mathrm{vol}((1 + {\epsilon'}^{-1})K_F + \Theta_F + M_F + {\epsilon'}^{-1}\Theta_F + (2 + {\epsilon'}^{-1})(2k+1)A_F) \\
\leq &\mathrm{vol}((1 + {\epsilon'}^{-1})(K_F + \Theta_F + P_F) + \delta^{-1}(-K_X|_F) \\
&+ m(2 + {\epsilon'}^{-1})(2k+1)(-K_X|_F)) \\
\leq &\mathrm{vol}(((1 + {\epsilon'}^{-1})(a-1) + \delta^{-1} + (2 + {\epsilon'}^{-1})(2k+1)m)(-K_X|_F)),
\end{aligned}$$

where A_F and Σ_F are the strict transforms of $A_{F'}$ and $\Sigma_{F'}$ on F respectively. Hence by Step 2, $\mathrm{vol}(K_{F'} + \Sigma_{F'} + 2(2d+1)A_{F'}) \leq v_1$ for a number v_1 depending only on d, ϵ, and δ.

Now we may apply Lemma 3.1 of Chapter 4 to

$$(Y, C, D, Z, H_Z) = (F, \mathrm{Supp}\,\Theta_F, m\delta M_F, F', A_{F'})$$

to construct a log bounded family $\overline{\mathcal{P}}$ of couples such that there exists a couple $(\overline{F}, \Sigma_{\overline{F}}) \in \overline{\mathcal{P}}$ satisfying the following.

- F is birational to $\overline{F}$, $(\overline{F}, \Sigma_{\overline{F}})$ is log smooth,

- $\Sigma_{\overline{F}}$ consists of the support of $M_{\overline{F}} + \Theta_{\overline{F}}$ and divisors exceptional over F,
- the coefficients of $M_{\overline{F}}$ are bounded from above by a number, say u, depending only on d, ϵ, and δ.

Here we may take a higher model of F' such that the induced map $g : F' \dashrightarrow \overline{F}$ is a morphism, $M_{\overline{F}} := g_*f^*M_F$, and $\Theta_{\overline{F}} := g_*f^*\Theta_F$.

Step 4. Apply Proposition 2.1 of Chapter 4.

We may assume that $a < 1$. Then $-(K_X + \Delta) \sim_{\mathbb{Q}} -(1-a)K_X$ is ample. By Shokurov-Kollár connectedness principle [**4**, 17.4], the non-klt locus of (X, Δ) is connected. By the construction, since x is not contained in $\mathrm{Supp}(\Delta_1)$, (X, Δ) is not lc at y' with a non-klt center not equal to G. By Shokurov-Kollár connectedness principle again, (X, Δ) has a non-klt center intersecting G, but not equal to G. By [**1**, Lemma 3.14], we can choose $P_F \geq 0$ such that (F, Δ_F) is not $\frac{\epsilon}{2}$-lc. Since (F, Λ_F) is sub-ϵ-lc, $(F, \Delta_F + (\Delta_F - \Lambda_F))$ is not klt by the discrepancy computation.

Now since (X, B) is ϵ-lc and G is general, by [**1**, Lemma 3.12] (see Section 2 of Chapter 9), there is a sub-boundary B_F on F such that (F, B_F) is sub-ϵ-lc, $K_F + B_F = (K_X + B)|_F \sim_{\mathbb{Q}} 0$ and $B_F = \Lambda_F + B|_F$. Moreover,

$$B_F + 2(\Delta_F - \Lambda_F) = \Delta_F + (\Delta_F - \Lambda_F) + B|_F,$$

and hence $(F, B_F + 2(\Delta_F - \Lambda_F))$ is not klt. In addition, $K_F + B_F + 2(\Delta_F - \Lambda_F) \sim_{\mathbb{Q}} -2aK_X|_F$ is ample. Write

$$K_{\overline{F}} + B_{\overline{F}} := g_*f^*(K_F + B_F).$$

Then by Exercise 6.4 of Chapter 4, $(\overline{F}, B_{\overline{F}} + 2g_*f^*(\Delta_F - \Lambda_F))$ is not sub-klt and by Exercise 4.1, $(\overline{F}, B_{\overline{F}})$ is sub-ϵ-lc.

Since $\Lambda_F \leq \Theta_F$, we have

$$\mathrm{Supp}(B_F^{\geq 0}) \subset \mathrm{Supp}(\Theta_F + B|_F) = \mathrm{Supp}(\Theta_F + M_F)$$

which implies that $\mathrm{Supp}(B_{\overline{F}}^{\geq 0}) \subset \Sigma_{\overline{F}}$. Applying Proposition 2.1 of Chapter 4 to the sub-pair $(\overline{F}, B_{\overline{F}})$, $L = 2g_*f^*(\Delta_F - \Lambda_F)$ and $\tilde{L} = 2a\delta M_{\overline{F}} \sim_{\mathbb{Q}} L$, there is a positive real number $\lambda \in \mathbb{R}_{>0}$ depending only on $\epsilon, \overline{\mathcal{P}}$ such that $2a\delta u > \lambda$, which implies that $a > \frac{\lambda}{2u\delta}$. □

3. Proof of the special BAB

The following special case of [**3**, Theorem 1.3] will be used to reduce Theorem 1.1 to the Fano case.

THEOREM 3.1 ([**3**, Theorem 1.3]). Let $d, m \in \mathbb{N}$. Consider projective varieties X satisfying:

- (X, B) is klt of dimension d for some boundary B,
- B is big and $K_X + B \sim_{\mathbb{Q}} 0$, and
- mB is an integral Weil divisor.

Then the set of such X forms a bounded family.

According to the following theorem, in order to prove Theorem 1.1, it suffices to show the log birational boundedness.

THEOREM 3.2 ([**2**, Theorem 1.6]). Let $d \in \mathbb{N}$ and $\delta, \epsilon \in \mathbb{R}_{>0}$. Consider the set of log pairs (X, Δ) such that

- (X, Δ) is ϵ-lc,

- $K_X + \Delta$ is ample, and
- the coefficients of Δ are at least δ.

If the set of such (X, Δ) is log birationally bounded, then it is log bounded.

Proof of Theorem 1.1. Let $X' \to X$ be a $\mathbb{Q}$-factorialization. Running a $(-K_{X'})$-MMP, we get a $(-K)$-minimal model $X_{\min}$. Since B is big, $-K_{X_{\min}}$ is nef and big. By the Basepoint-free Theorem (cf. [**7**, Theorem 3.3]), $-K_{X_{\min}}$ is semiample, and defines a birational contraction $X_{\min} \to X_{\text{can}}$, where $-K_{X_{\text{can}}}$ is ample. Denote by B_{can} the birational transform of B on X_{can}. Note that $K_{X_{\text{can}}} + B_{\text{can}} \sim_{\mathbb{R}} 0$ and $(X_{\text{can}}, B_{\text{can}})$ is again ϵ-lc by Exercise 4.1.

If the set of such X_{can} is bounded, then there is a natural number n, such that $-nK_{X_{\text{can}}}$ is Cartier (cf. Exercise 4.4). Moreover, by the Effective Basepoint-free Theorem [**5**], we may assume that $-nK_{X_{\text{can}}}$ is base point free. This implies that there is a klt n-complement of $K_{X_{\text{can}}}$, which in turn gives a klt n-complement of K_X (cf. Exercise 4.5). Hence by Theorem 3.1, X forms a bounded family.

Finally, in order to show the boundedness of X_{can}, we show that $(X_{\text{can}}, B_{\text{can}})$ forms a log bounded family. We may take $\Delta_{\text{can}} = (1+t)B_{\text{can}}$ for some sufficiently small $t > 0$, such that $(X_{\text{can}}, \Delta_{\text{can}})$ is $\frac{\epsilon}{2}$-lc where $K_{X_{\text{can}}} + \Delta_{\text{can}} \sim_{\mathbb{Q}} -tK_{X_{\text{can}}}$ is ample. By Theorem 3.2, it suffices to prove that $(X_{\text{can}}, \Delta_{\text{can}})$ forms a log birationally bounded family, or equivalently, $(X_{\text{can}}, B_{\text{can}})$ is log birationally bounded. By Proposition 4.2 of Chapter 4 and Proposition 1.2, there exist $m \in \mathbb{N}$ and v depending only on d, ϵ, and δ, such that $|-mK_{X_{\text{can}}}|$ defines a birational map and $\text{vol}(-K_{X_{\text{can}}}) < v$. Take $\phi : W \to X_{\text{can}}$, A_W, R_W as in Notation 3.2 of Chapter 4. Write Σ_W to be the support of $\phi_*^{-1}B_{\text{can}}$ and all ϕ-exceptional divisors, then

$$\begin{aligned}
&\text{vol}(K_W + \Sigma_W + 2(2d+1)A_W) \\
\le &\text{vol}(K_{X_{\text{can}}} + \phi_*\Sigma_W + 2(2d+1)\phi_*A_W) \\
\le &\text{vol}(K_{X_{\text{can}}} + \delta^{-1}B_{\text{can}} + 2(2d+1)\phi_*A_W) \\
\le &\text{vol}(-(\delta^{-1} + (4d+2)m)K_{X_{\text{can}}}) \\
\le &(\delta^{-1} + (4d+2)m)^d v.
\end{aligned}$$

Now we may apply Lemma 3.1 of Chapter 4 to

$$(Y, C, D, Z, H_Z) = (X_{\text{can}}, 0, mB_{\text{can}}, W, A_W)$$

to finish the proof. □

4. Exercises

Exercise 4.1. *Let (X, B) be an ϵ-lc pair and $K_X + B \equiv 0$. If $g : W \to X, h : W \to Y$ are two birational contractions, then $h_*g^*(K_X + B)$ is sub ϵ-lc. In particular, if $f : X \dashrightarrow X'$ is a birational contraction, then (X', f_*B) is also ϵ-lc.*

Exercise 4.2. *Let $f : X \dashrightarrow Y$ be a D-non-positive birational contraction between normal projective varieties. Show that* $\text{vol}(D) = \text{vol}(f_*D)$.

Exercise 4.3. *Let (X, B) be an ϵ-lc pair such that $K_X + B \equiv 0$. Then for any $\epsilon' \in (0, \epsilon)$, we may find a $\mathbb{Q}$-divisor B', such that (X, B') is an ϵ'-lc pair and $K_X + B' \sim_{\mathbb{Q}} 0$.*

Exercise 4.4. *Suppose that X belongs to a bounded family $\mathcal{P}$ and X is projective with klt singularities. Then there is a natural number n depending on $\mathcal{P}$, such that nK_X is Cartier.*

EXERCISE 4.5. *Let $f : X \dashrightarrow Y$ be a $(-K_X)$-non-positive birational contraction between normal projective varieties. If K_Y has a klt n-complement for some natural number n, show that K_X also has a klt n-complement.*

References

1. Caucher Birkar, *Anti-pluricanonical systems on fano varieties*, **123** (2016).
2. Christopher D. Hacon, James McKernan, and Chenyang Xu, *ACC for log canonical thresholds*, Ann. of Math. (2) **180** (2014), no. 2, 523–571. MR 3224718
3. Christopher D. Hacon and Chenyang Xu, *Boundedness of log Calabi-Yau pairs of Fano type*, Math. Res. Lett. **22** (2015), no. 6, 1699–1716. MR 3507257
4. János Kollár, *Flips and abundance for algebraic threefolds*, Lectures in Utah Summer Seminar 1991, Asterisque, vol. 211, 1992.
5. ———, *Effective base point freeness*, Math. Ann. **296** (1993), no. 4, 595–605. MR 1233485
6. ———, *Singularities of pairs*, Algebraic geometry—Santa Cruz 1995, Proc. Sympos. Pure Math., vol. 62, Amer. Math. Soc., Providence, RI, 1997, pp. 221–287. MR 1492525
7. János Kollár and Shigefumi Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998, With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original. MR 1658959

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